The Critical Range in Clustered Ad Hoc Networks: An Analysis for Gaussian Distributed Nodes

Maria Michalopoulou, *Student Member, IEEE*, and Petri Mähönen, *Senior Member, IEEE*

***Abstract*—Connectivity is one of the essential properties of wireless networks. Although it is influenced by many environmental factors, it exhibits a strong and fundamental dependence on the distances between the nodes. Therefore, the connectivity of wireless networks from distance perspective has been a subject of constant interest for the research community. However, despite the fact that the assumption of uniformly distributed nodes is highly unrealistic for commercial wireless networks, analytical work on the connectivity of non-uniform node distributions is almost non-existent. In many real cases wireless nodes may be placed in a way such that they form clusters. In this letter we consider a set of clustered nodes distributed according to a symmetric Gaussian distribution and we present an approximate estimation of the probability density function of the critical range of the cluster. This is, to the best of our knowledge, the first estimation of the critical range for a clustered distribution.**

***Index Terms*—Clustered node locations, Gaussian spatial distribution, critical range.**

I. Introduction

THE connectivity of a wireless ad hoc network is fundamentally dependent on the physical distances between the nodes. Due to environmental factors, such as interference and fading, in reality the received power is not a deterministic and monotonically decreasing function of the distance; however, this does not eliminate the strong dependency of the received power on the distance. Thus the critical transmission range is a fundamental parameter for studying connectivity of wireless ad hoc networks. In terms of connectivity from distance perspective, a network is connected (i.e., there exists a path between any two nodes) if communication between any two nodes is guaranteed for separation distances smaller than or equal to the critical range.

There is a plethora of analytical studies on the connectivity of wireless networks. One of the best known studies is by Gupta and Kumar [1] who derived results on the critical transmission power required for guaranteeing network connectivity. The critical density of nodes at which percolation occurs is a question widely discussed in the literature [2], [3]. Furthermore, several studies address the problem from the angle of estimating the number of neighbors needed for ensuring network connectivity [4], [5]. The network connectivity from percolation perspective has also relevance in the context of stochastic geometry [6].

Manuscript received October 23, 2014; revised February 9, 2015; accepted February 10, 2015. Date of publication February 24, 2015; date of current version May 7, 2015. The associate editor coordinating the review of this paper and approving it for publication was V. Eramo.

The authors are with the Institute for Networked Systems, RWTH Aachen University, 52062 Aachen, Germany (e-mail: mmi@inets.rwth-aachen.de; pma@inets.rwth-aachen.de).

Digital Object Identifier 10.1109/LCOMM.2015.2406318

The vast majority of research work on the topic is conducted on topologies of randomly distributed nodes, sampled from a Poisson spatial distribution. However, over the past years it has been recognized that the assumption of uniformly distributed nodes is rather implausible for real, deployed wireless networks. And in fact, it has been also shown that the spatial distribution of the nodes has a significant impact on several properties of the network [7]–[9], among which is connectivity. Therefore, connectivity properties of uniformly distributed nodes could be used, at best, as an approximation reference. Despite this, as far as we are aware, there are only few studies that go beyond the assumption of random uniform node distributions. For instance, in [10], Bettstetter assumes a certain transmission range and derives generic probabilistic results on connectivity for some spatial probability density function. In [11] the authors generate an inhomogeneous Poisson process and present stochastic results on the nearest neighbor distances. In [7] and [8] the authors demonstrate some simulation analyses on the connectivity properties of several non-uniform point distributions. Santi *et al.* [12] consider Poisson distributed nodes, but present also simulations on the effect of mobility on the network connectivity.

In this letter, considering a clustered set of nodes placed according to a symmetric Gaussian distribution, we perform an approximate estimation of the probability density function (pdf) of the critical transmission range. In many real cases the distribution of wireless nodes exhibits some sort of spatial clustering, thus clustered distributions are generally of practical interest. Gaussian clusters can be also a reasonable toy model for local area wireless networks [13]. This work makes two contributions. The first is the estimation of the critical transmission range for symmetric Gaussian cluster. Second, it aims to suggest a generic approach that can be used for addressing other types of clustered distributions as well.

II. Analytical Estimation of the Critical Range

We consider a set of wireless nodes forming a Gaussian cluster. That is, given a location as the center of the cluster we consider N wireless nodes placed around the cluster center according to a two-dimensional Gaussian spatial distribution. Our aim is to estimate the critical transmission range for this set of nodes.

Without loss of generality we assume that the center of the cluster is at the origin of the coordinate system, i.e., at point (0, 0). The locations of the nodes around the cluster center are generated by a bivariate Gaussian distribution with mean $\boldsymbol{\mu} = [0, 0]$ and a diagonal covariance matrix $\Sigma = \mathrm{diag}(\sigma_x^2, \sigma_y^2)$. The pdf describing the probability that an arbitrary point of the Gaussian point process (i.e., a node) is located at coordinates (x, y) is given by

$$p_{xy} = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right). \quad (1)$$

Considering a symmetric Gaussian distribution with $\sigma_x = \sigma_y = \sigma$, the p_{xy} depends only upon the distance of the location (x, y) from the cluster center. Equation (1) reduces to

$$p_{xy} = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{2}$$

where r denotes the distance of the location (x, y) from the cluster center (that is, $r^2 = x^2 + y^2$). Thus, Equation (2) expresses the pdf that an arbitrary node lies at a location whose distance from the cluster center is equal to r.

There will be a number of outliers lying exceptionally away from the cluster center and at relatively large distances from all the other nodes in the cluster. The critical range of the cluster will be most likely imposed by the most distant outlier node, that is, the node lying at the largest distance from the cluster center. For the sake of shortness we shall call this node the *outlier node*. Therefore, we consider the critical range to be the distance between the outlier node and its nearest neighbor. The validity of this statement can be easily justified. For every point in the cluster, P, let us define the radius R_P of the circle centered at P so that the circle (P, R_P) is expected to contain exactly one node apart from P, i.e., the nearest neighbor of P. The expected number of points, apart from P, that fall in the circle (P, R_P) is given by the following integral

$$(N-1) \iint\limits_{(P,R_P)} p_{xy} \mathrm{d}A, \tag{3}$$

where the area of integration (P, R_P) is the circle with radius R_P centered at P and $\mathrm{d}A$ represents an infinitesimal area element. The assumption that the critical range is imposed by the outlier point is equivalent to the following statement: For any two points, P_1 and P_2, at distances r_1 and r_2, respectively, from the cluster center, the inequality $R_{P_1} > R_{P_2}$ holds if and only if $r_1 > r_2$. By means of Equations (2) and (3) we can write the following equality

$$\iint\limits_{(P_1,R_{P_1})} \exp\left(-\frac{r^2}{2\sigma^2}\right) \mathrm{d}A \overset{!}{=} \iint\limits_{(P_2,R_{P_2})} \exp\left(-\frac{r^2}{2\sigma^2}\right) \mathrm{d}A. \tag{4}$$

Since the integrand at both sides of Equation (4) is a monotonically decreasing function of the distance from the cluster center, for the equality to hold the radius R_{P_1} needs to be larger than R_{P_2} if $r_1 > r_2$. Therefore, the outlier point is expected to determine the critical range. In fact, this statement holds for every point distribution where the probability that an arbitrary point lies at a certain location is monotonically decreasing with the distance from the cluster center.

Let us refer to the distance of the outlier node from the cluster center as the *outlier distance*. We derive the probability that the outlier distance is approximately equal to r_o, namely between $r_o - \epsilon$ and $r_o + \epsilon$ (with $\epsilon \ll r_o$), by calculating the probability that there is one arbitrary node located at a distance equal to $r_o \pm \epsilon$, whereas all the other nodes are not farther than $r_o + \epsilon$ from the cluster center. Since the locations of the nodes are independent from each other, the probability that the outlier distance in the cluster is approximately r_o (that is, $r_o \pm \epsilon$) is derived as follows

$$p_{r_o} = N\, p[0 < r \leq r_o + \epsilon]^{N-1}\, p[r_o - \epsilon \leq r \leq r_o + \epsilon]. \tag{5}$$

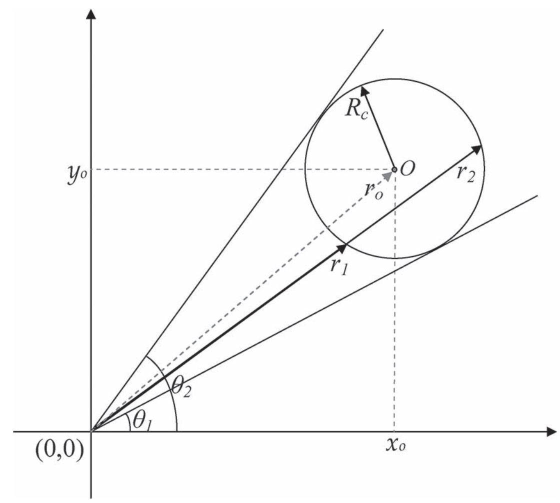

Fig. 1. The geometry of the problem. The origin of the coordinate system is the cluster center and point O is the outlier node. The critical range R_c is calculated such that exactly one node is expected to be found within the circle (O, R_c).

The quantity $p[r' \leq r \leq r'']$ is the probability that the distance of an arbitrary node from the cluster center lies between r' and r'' and can be calculated by means of Equation (2) as follows

$$p[r' \leq r \leq r''] = \int_0^{2\pi}\int_{r'}^{r''} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{6}$$

As discussed above, we expect that the critical range of the cluster is given by the distance between the outlier node and its closest node in the cluster. Therefore, for a random value of the outlier distance, let us say r_o, we have to estimate how far from the outlier node we expect to find the closest node. In other words, we want to find the radius of the smallest possible disc centered at the outlier node, such that exactly one other node is expected to be found within the disc. The geometry of the problem is illustrated in Fig. 1. The cluster center is at the origin of the coordinate system and the outlier node with coordinates x_o and y_o, call it O, is lying at distance r_o from the cluster center. We want to find the radius, R_c, of the circle centered at O, so that exactly one node (apart from the outlier node itself) is expected to fall within the circle.

The probability that an arbitrary node is lying within the circle (O, R_c), i.e., within distance R_c from the outlier node, is given by the following polar integral

$$p_{R_c} = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\, \mathrm{d}r\, \mathrm{d}\theta, \tag{7}$$

where the limits of integration for θ and $r(\theta_1, \theta_2, r_1, r_2)$ are depicted in Fig. 1. The angles θ_1 and θ_2 are the slopes of the two tangent lines to the circle (O, R_c) that pass through the cluster center. The distances r_1 and r_2 are expressed as functions of the angle θ. We consider the line that crosses the circle (O, R_c) and passes through the cluster center, that is, the line with equation $y = \tan(\theta)x$, and we find the two points where the line intersects the circle (O, R_c). The distances of the two intersection points from the cluster center are the distances r_1 and r_2.

TABLE I
ALGORITHM FOR THE ESTIMATION OF R_c

$r_c \leftarrow 0$ ▷ expected neighbors of O in circle (O, r_c)
$N_{r_c} \leftarrow 0$ ▷ expected neighbors of O in circle (O, r_c)
while $N_{r_c} < 1$ **do**
  $r_c \leftarrow r_c + d_{r_c}$ ▷ where d_{r_c} is a small distance step
  ▷ Equation of circle (O, r_c)
  A: $(y - y_o)^2 + (x - x_o)^2 = r_c^2$
  ▷Equation of line passing through the cluster center $(0, 0)$
  B: $y = (\tan\theta)x$
  ▷ Finding the limits of integration for Equation 7 $(\theta_1, \theta_2, r_1, r_2)$
  Obtain Equation AB by eliminating y from the system of Equations A and B.
  Let Δ be the discriminant of Equation AB.
  Solve Equation $\Delta = 0$ with respect to θ and obtain two solutions, θ_1 and θ_2.
  Solve the system of Equations A and B with respect to x and y and obtain two solutions, $(x_1(\theta), y_1(\theta))$ and $(x_2(\theta), y_2(\theta))$.
  $r_1 \leftarrow \sqrt{x_1^2 + y_1^2}, \quad r_2 \leftarrow \sqrt{x_2^2 + y_2^2}$
  if $r_1 > r_2$ **then**
    Swap r_1 and r_2
  end if
  Calculate the probability, p_{r_c}, that a node lies within distance r_c from point O (from Equation 7).
  $N_{r_c} = (N-1)p_{r_c}$ ▷ analogously to Equation 8
end while
$R_c \leftarrow r_c$ ▷ set the critical range equal to r_c

Using Equation (7) we can estimate the expected number of nodes within distance R_c from the outlier node to be

$$E\left[N_{R_c}\right] = (N-1)p_{R_c}. \tag{8}$$

Thus, with probability p_{r_o} the critical range is equal to R_c, where R_c is calculated such that

$$E\left[N_{R_c}\right] \overset{!}{=} 1. \tag{9}$$

Given that, we are able to estimate the pdf of the critical range by estimating R_c for a range of values of r_o. A set of pairs of critical ranges and their corresponding probabilities (R_c and p_{r_o}) can be used properly for estimating the pdf of the critical range. Although the main result of the present letter is the mathematical derivation of the critical range, we provide also the algorithm for finding R_c for a specific value of the outlier distance in Table I. The algorithm aims to be useful to simulation groups in the wireless networks community interested in exploiting our result for simulation purposes.

We note that the estimation of R_c based on Equation (9) provides an expected value for R_c so that the outlier node is connected with its closest neighbor, however, it is not a guarantee that the outlier node is connected. The probability that the node is actually connected is $1 - (1 - p_{R_c})^{N-1}$, where $p_{R_c} = 1/(N-1)$ according to Equation (9). To further increase it, the goal can be set to a higher value, i.e., $E[N_{R_c}] \overset{!}{=} c > 1$.

We would like to point out that although a closed-form expression for the critical range would be desirable, it is not a straightforward task since it requires the estimation of the conditional probability that the critical transmission range is equal to r_c, given than the outlier point is at distance $r_o (p_{r_c|r_o})$. The outlier point is at distance approximately equal to r_o from the cluster center, but this does not necessarily hold only for a single node; this fact complicates the estimation of $p_{r_c|r_o}$. Furthermore, the outlier point exhibits the higher probability of being the one that shall determine the outlier distance, however, this is not a guaranteed fact. It is probable that the critical range is determined by a node at distance from the cluster center slightly smaller than the outlier distance, which means that even a calculation of the probability $p_{r_c|r_o}$ would not be sufficient for estimating the critical range. Nevertheless, the approach we proposed by means of pairs of values (p_{r_o}, R_c) is fairly accurate; even if the critical range is the distance between a node at distance slightly smaller than r_o and its nearest neighbor, the pair (p_{r_o}, R_c) is still a reasonable approximation.

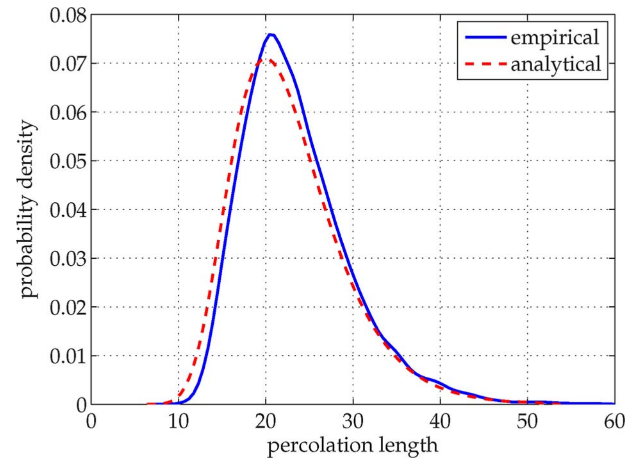

Fig. 2. The pdf of the critical range in a symmetric Gaussian cluster with 200 points and $\sigma = 20$.

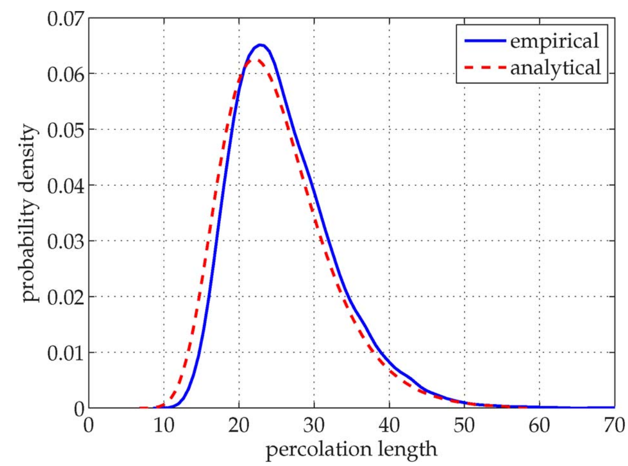

Fig. 3. The pdf of the critical range in a symmetric Gaussian cluster with 50 points and $\sigma = 20$.

III. SIMULATION RESULTS

To illustrate the validity of the derivation we described in Section II, we make a comparison with empirical results obtained by simulations. We generated symmetric Gaussian clusters with different number of nodes (N) and different values of standard deviation (σ) to verify our results. Additionally, the simulation results provide an insight on how much a limited sampling (Monte-Carlo sampling) might bias the results. For each generated random realization we run a clustering algorithm to find the smallest distance for which the network becomes connected. Figs. 2–4 show examples of the

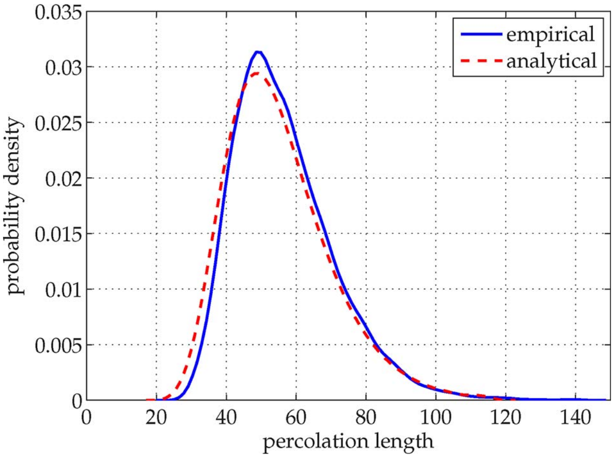

Fig. 4. The pdf of the critical range in a symmetric Gaussian cluster with 300 points and $\sigma = 50$.

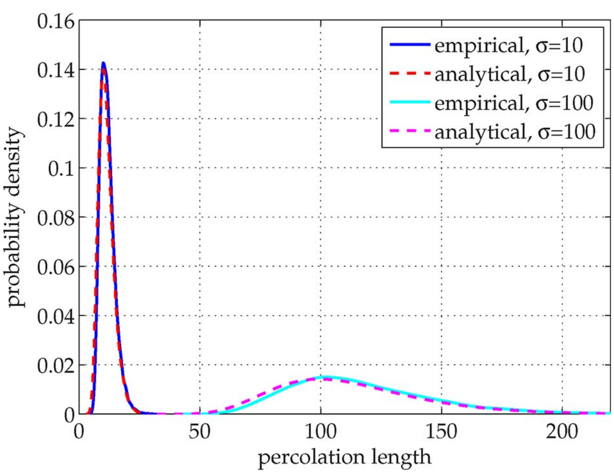

Fig. 5. The pdfs of the critical ranges in two symmetric Gaussian clusters with $\sigma = 10$ and $\sigma = 100$. In both cases the number of points is 200.

analytically estimated pdfs of the critical ranges together with the corresponding empirical pdfs. An empirical pdf of the critical range (for a specific value of N and a specific value of σ) was obtained after averaging the results of 10 000 random realizations.

It is interesting to note that when it comes to a spatial Gaussian distribution, the standard deviation, σ, affects strongly the average value of the critical range and the width of its pdf. This is illustrated in Fig. 5, where we show two pdfs of the critical ranges when the σ of the Gaussian cluster changes. One should also note that the derived pdf is not symmetric around the peak value, but exhibits a long tail towards large critical ranges.

IV. Conclusion

We presented an approximate estimation of the probability density function of the critical range in a symmetric Gaussian distributed cluster. Despite the fact that connectivity in terms of distance has been a widely researched topic in the context of wireless networks, the vast majority of research studies are based on the assumption of uniformly distributed wireless nodes. Nevertheless, it has been realized over the past years that other types of node distributions are more prominent in existing wireless networks. As far as we are aware, this is the first derivation of results on the critical range of a clustered distribution. Apart from the derivation presented for the specific distribution, namely the Gaussian, the approach we followed can be used for addressing other clustered distributions as well.

In this letter we considered wireless nodes that are spatially distributed forming a cluster, however, the case of multiple clusters of wireless nodes is, naturally, an open question of interest. As a future work we are working on the analytical estimation of the percolation length between clusters.

References

[1] P. Gupta and P. Kumar, "Critical power for asymptotic connectivity in wireless networks," in *Stochastic Analysis, Control, Optimization and Application*, W. M. McEneaney, G. Yin, and Q. Zhang, Eds. Boston, MA, USA: Birkhäuser, 1999, pp. 547–566.

[2] M. D. Penrose, "On a continuum percolation model," *Adv. Appl. Probability*, vol. 23, no. 3, pp. 536–556, Sep. 1991.

[3] S. C. Ng, G. Mao, and B. Anderson, "Critical density for connectivity in 2D and 3D wireless multi-hop networks," *IEEE Trans. Wireless Commun.*, vol. 12, no. 4, pp. 1512–1523, Apr. 2013.

[4] F. Xue and P. R. Kumar, "The number of neighbors needed for connectivity of wireless networks," *Wireless Netw.*, vol. 10, no. 2, pp. 169–181, Mar. 2004.

[5] C. Bettstetter, "On the minimum node degree and connectivity of a wireless multihop network," in *Proc. 3rd ACM Int. Symp. MobiHoc*, 2002, pp. 80–91.

[6] M. Haenggi, *Stochastic Geometry for Wireless Networks.* Cambridge, U.K.: Cambridge Univ. Press, 2012.

[7] M. Petrova, P. Mähönen, and J. Riihijärvi, "Connectivity analysis of clustered ad hoc and mesh networks," in *Proc. IEEE GLOBECOM*, Nov. 2007, pp. 1139–1143.

[8] J. Riihijärvi, M. Petrova, and P. Mähönen, "Influence of node location distributions on the structure of ad hoc and mesh networks," in *Proc. IEEE GLOBECOM*, Nov. 2008, pp. 1–5.

[9] R. Ganti and M. Haenggi, "Interference and outage in clustered wireless ad hoc networks," *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 4067–4086, Sep. 2009.

[10] C. Bettstetter, "On the connectivity of ad hoc networks," *Comput. J.*, vol. 47, no. 4, pp. 432–447, 2004.

[11] C. Bettstetter, M. Gyarmati, and U. Schilcher, "An inhomogeneous spatial node distribution and its stochastic properties," in *Proc. 10th ACM Symp. MSWiM*, 2007, pp. 400–404.

[12] P. Santi and D. Blough, "The critical transmitting range for connectivity in sparse wireless ad hoc networks," *IEEE Trans. Mobile Comput.*, vol. 2, no. 1, pp. 25–39, Jan. 2003.

[13] J. Riihijärvi, P. Mähönen, and M. Rübsamen, "Characterizing wireless networks by spatial correlations," *IEEE Commun. Lett.*, vol. 11, no. 1, pp. 37–39, Jan. 2007.