Modeling Spatial Structure of Wireless Communication Networks

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***Abstract*—While modeling and analysis of network topology has been an active area of research in fixed networks, much less work has been done towards realistic modeling of wireless networks. The graph-based approach that has served as solid foundation for network science in the fixed domain is not natural for wireless communication networks, since their performance inherently depends on the *spatial* relationships between nodes. In this paper we apply techniques from spatial statistics literature to develop models of the spatial structure of the network for a variety of wireless network types. In particular, we construct models of television and radio transmitter distributions that have applications in, for example, cognitive wireless network applications. We use a stochastic approach based on fitting parametric location models to empirical data. Our results indicate that the so-called *Geyer saturation model* can accurately reproduce the spatial structure of a large variety of wireless network types, arising from both planned or chaotic deployments. The resulting models can be used in simulations or as basis of analytical calculations of different network properties, and we believe that the presented methodology can serve as a solid foundation for the emerging network science of wireless communication networks.**

I. Introduction

Key research activity in the context of both wireless and fixed networks has been the development of *realistic* and *validated* stochastic models for phenomena relating to those networks. Examples of well-developed models are those of, e.g., wireless channels, mobility and traffic patterns. A substantially less well developed area is that of modeling the network structure itself. In the fixed network domain there has been significant activities towards modeling the topology and structure of the network in terms of its connectivity structure (see, for example, [1], [2], [3]), but in wireless networks research where actual node locations play key role much less has been done. By far the most common assumption still used in both simulation work and analytical calculations is that nodes of a wireless network are uniformly distributed over some region $E \subseteq \mathbb{R}^2$. Some examples of work *not* making this assumption are [4], [5], [6], in which the impact of deviations from uniformity is also discussed. However, even in those references the used node location models are motivated only by having a qualitative difference to the uniform case. Another extreme approach has been to use actual node locations from existing networks in simulations. This guarantees certain level of realism, but does not allow general conclusions to be drawn. Thus, from the wireless networking point of view, one of the key challenges for network science is to come up with right abstractions and models for spatial structure of wireless networks that retains their central structural characteristics, but are general enough to enable applications in analysis and stochastic simulation.

In this paper we make a first attempt at developing *realistic models of node locations* for wireless networks research. We use techniques from stochastic geometry [7], [8] and spatial statistics [9], [10] to fit stochastic location models to a variety of data sets, and validate the goodness-of-fit using correlation statistics [11]. Spatial statistics techniques have been applied extensively in a number of fields such as geography, physics and biology to characterize and model spatial structure of diverse phenomena. However, applications to wireless networks are almost non-existent in the literature. Stochastic geometry on the other hand has been applied extensively to wireless networking problems to derive theoretical results on, for example, distribution of interference, typical distances between nodes, or topological structure of the network once a simple model for connectivity has been assumed.

The key limitation in this line of work is that usually rather restrictive assumptions, such as uniformly random distribution of node locations, are needed for the analytical calculations to become feasible. This also limits the extent experimental data gathered from the structure of different networks can be incorporated into the calculations. For the uniform random case, basically only the average number of nodes in a given area can be matched to experimental data, and indeed we are not aware of any prior work that takes the stochastic geometry approach beyond this. Finally, it should be noted that the graph-based topology characterization techniques that have successfully been applied to fixed networks have been used to analyze wireless networks as well. However, such analysis can easily lead into highly misleading results. This is due to the fundamental characteristic of wireless communications, which is inherently spatial in nature. Due to factors such as mutual interference, power control, noise, and fluctuations in propagation conditions graph-based representation of a wireless network often results in a too high level of abstraction.

By adopting the spatial statistics approach, we seek to develop models that can reproduce key characteristics of the structure of a wireless network, but still have a solid theoretical foundation. The modeling of spatial structure of the network enables also more complex analysis of wireless networks by combining the models developed here with already existing

models of propagation, user behavior, interference etc. The results show that a relatively simple parametric model can accurately match a variety of network types, resulting from both planned and chaotic deployments. We give a thorough introduction to the methodology and underlying theory, and then give selected case studies of the fitting process itself. Since we are discussing ongoing work, we also briefly outline our current and future work on studying location models developed considering the network deployment process itself, and how location models can be applied in problems related to dynamic spectrum access networks [12]. We believe that the techniques presented can serve as one foundational component in the developing field of network science, especially when dealing with problems in which the spatial structure of the network in question plays a central role.

The rest of the paper is structured as follows. In Section II we give a short overview of theory of point processes, including key statistics that can be used in modeling work driven by experimental data, and selected models. In Section III we focus on the problem of fitting a parametric point process model on empirical data by means of maximum pseudolikelihoods, and how the resulting fits can be validated by suitable statistical analysis. Examples of fits obtained for actual node location data sets are discussed in Section IV, and future work and applications are given in Section V. Finally, in Section VI conclusions are drawn.

II. Node Locations as Point Processes

Informally, point processes yield randomly distributed collections of indistinguishable locations $\{X_i\}_{i=1}^n$ in some regular enough region E. Both the individual locations $X_i \in E$ and the total number of points n are in general random, but not independent of each other. Formally, a point process is defined as a random counting measure N, assigning to each measurable set $A \subseteq E$ a random variable $N(A)$, the number of points in A [13], [7]. One can always write

$$N = \sum_{i=1}^{n} \varepsilon_{X_i}, \tag{1}$$

where ε_x is a point mass at x (the measure equivalent to Dirac delta distribution), the X_i are E-valued random variables, and n is a random variable with values in the set of extended natural numbers (i.e., in $\mathbb{N} \cup \{+\infty\}$). Usually N is assumed to be *simple*, meaning that the points X_i are almost surely distinct.

The simplest example of a point process is that of the *homogeneous Poisson point process*, for which $N(A)$ is Poisson distributed with parameter $\nu|A|$, where $\nu \in \mathbb{R}_+$ and $|A|$ denotes the Lebesgue measure of A (in the two-dimensional case $|A|$ is simply the area of A). Additionally, $N(A)$ and $N(B)$ are required to be independent for any disjoint A and B. We can obtain richer models (called *Gibbs processes*) by specifying a *probability density* of a point process N, defined as the Radon-Nikodym derivative f of N with respect to homogeneous Point process with $\nu = 1$. For example, the density function for the homogeneous Poisson point process of arbitrary intensity ν is simply

$$f(\boldsymbol{X}) = \exp((1-\nu)|E|)\nu^{\#(\boldsymbol{X})}, \tag{2}$$

where $\boldsymbol{X} = \{X_1, \ldots, X_n\}$, with $X_i \in E$, and $\#(\boldsymbol{X}) = n \geq 0$.

The specific model we apply in the following is the *Geyer saturation process* [14], a generalisation of the Strauss process [15]. For the Strauss process we have the density

$$f(\boldsymbol{X}) = \alpha\beta^{\#(\boldsymbol{X})}\gamma^{s_r(\boldsymbol{X})}, \tag{3}$$

where $s_r(\boldsymbol{X})$ denotes the number of point pairs of $\boldsymbol{X}$ that are closer than distance r apart, $\beta > 0$ and $0 \leq \gamma < 1$ (α is for normalization). The Strauss model is powerful tool for modeling *regular* point processes, in which individual points tend to be separated by some minimum distance, but it cannot model *clustered* distributions of locations. For the Geyer process an additional *saturation threshold* ζ is added, bounding the contribution of the exponent of γ. The case $\zeta \to \infty$ yields the Strauss process as a limit. The advantage of the Geyer process is that it removes the limitation $0 \leq \gamma < 1$ of the Strauss process, enabling both clustered (for which $\gamma > 1$) and regular processes to be modeled. Notice that if $\gamma = 1$, both the Strauss and Geyer processes reduce to the homogeneous Poisson point process.

We shall conclude this section by defining statistics for point processes used for evaluating goodness of fit in the following. Due to space reasons we shall be rather brief, and refer the reader to [16] to more detailed discussion and examples. We shall specialize to the case $E = \mathbb{R}^n$, and assume that N is stationary, that is, that the structure of N remains invariant under translations. Under these conditions the *mean measure* $\mu_N(A) \equiv \mathbb{E}\{N(A)\}$, yielding the average number of points in A, is simply $\nu|A|$, where the constant ν is the *intensity* of N. Using the Euclidean metric we can study the cumulative distribution functions of distances to nearest point of N from an arbitrary point of E, or from another point of N. The latter results in the *nearest neighbor distance distribution function* $G(r)$, and the former yields the *empty space function* $F(r)$. These can be combined to the J-function by $J(r) \equiv (1-G(r))/(1-F(r))$. The interpretation of $J(r)$ is simple: values above one indicate regularity, and values below one indicate clustering. For the homogeneous Poisson point process we have $J(r) = 1$.

The simplest second-order statistic we shall consider is the *Ripley K-function* defined heuristically by

$$K(r) \equiv \frac{1}{\nu}\mathbb{E}\left\{\#(\boldsymbol{X} \cap B(x,r) - \{x\}) \mid x \in \boldsymbol{X}\right\}, \tag{4}$$

where $B(x,r)$ denotes the ball of radius r centered at x. For homogeneous Poisson point process we have in $E = \mathbb{R}^2$ that $K(r) = \pi r^2$, and in general $\nu K(r)$ is the expected number of points in $B(x,r)$ for $x \in \boldsymbol{X}$. We shall also use the L-function defined by $L(r) \equiv \sqrt{K(r)/\pi}$ below, since it is slightly easier to interpret, uniform distribution of locations yielding the straight line $L(r) = r$. For clustered distributions

of locations $L(r) > r$, whereas $L(r) < r$ indicates regularity in the corresponding length scales. In general second order statistics such as the K-function are more powerful than statistics based on nearest neighbor distances, and we shall reflect this difference in our applications below as well.

III. Fitting Point Processes to Empirical Data

For fitting point processes to data we shall primarily use a parametric approach. That is, we select a family of point processes $\{N_{\boldsymbol{\theta}}\}$ parameterised by $\boldsymbol{\theta}$, which we seek to select so that the statistics of $N_{\boldsymbol{\theta}}$ in some sense correspond to those of the empirical process obtained from the data. Classically the most appropriate technique for fitting a stochastic model to data is the method of maximum likelihood. Unfortunately maximum likelihood estimation is in general infeasible or computationally intensive for point processes, and pseudolikelihoods are often used instead. This will also be the case in the following. We focus on Gibbs processes since for them pseudolikelihoods can often be explicitly computed, and the techniques for evaluating the goodness of the fit obtained are relatively well understood. We thus assume a process with a parameterised density function $f_{\boldsymbol{\theta}}(\boldsymbol{X})$, for which the *Papangelou conditional intensity* is defined by

$$\lambda_{\boldsymbol{\theta}}(x, \boldsymbol{X}) \equiv \frac{f_{\boldsymbol{\theta}}(\boldsymbol{X} \cup x)}{f_{\boldsymbol{\theta}}(\boldsymbol{X})} \tag{5}$$

for points $x \notin \boldsymbol{X}$, and by

$$\lambda_{\boldsymbol{\theta}}(x, \boldsymbol{X}) \equiv \frac{f_{\boldsymbol{\theta}}(\boldsymbol{X})}{f_{\boldsymbol{\theta}}(\boldsymbol{X} - x)} \tag{6}$$

otherwise. The *log-pseudolikelihood* then becomes

$$\log \mathrm{PL}(\boldsymbol{\theta}, \boldsymbol{X}) \equiv \sum_{x_i \in \boldsymbol{X}} \log \lambda_{\boldsymbol{\theta}}(x_i, \boldsymbol{X}) - \int_E \lambda_{\boldsymbol{\theta}}(x, \boldsymbol{X})\, dx. \tag{7}$$

Finding the extremum of $\log \mathrm{PL}(\boldsymbol{\theta}, \boldsymbol{X})$ with respect to $\boldsymbol{\theta}$ defines then the fitted model with conditional intensity $\hat{\lambda}(x, \boldsymbol{X})$ evaluated for the data. This can in turn be used to define the *residual*

$$R(B) \equiv \#(B \cap \boldsymbol{X}) - \int_B \hat{\lambda}(x, \boldsymbol{X})\, dx \tag{8}$$

with applications in testing the goodness of fit [17], [18]. The actual maximisation of the log-pseudolikelihood can be performed using the numerical techniques developed by Baddeley and Turner [19] and implemented in [20].

IV. Case Studies and Example Results

We shall now illustrate the application of the above techniques on selected data sets. We shall first consider the access point locations of the Google Mountain View Wi-Fi network, which have been made publicly available [21]. Our treatment of this example will be fairly detailed, outlining the necessary steps from initial study of the data set to model selection, fitting and validation. We shall later give results for other network types as well in a more concise manner. For carrying out the calculations involved we use the spatstat software suite [20].

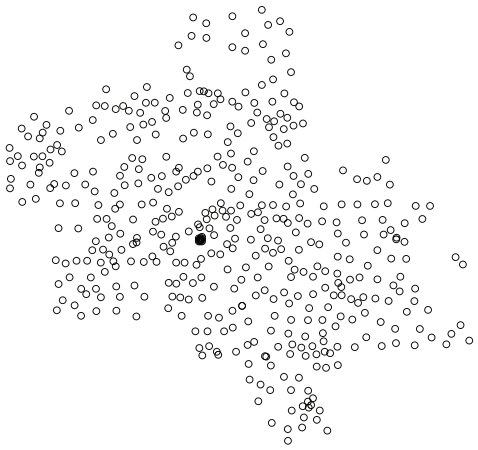

Fig. 1. Data set consisting of the access point locations of the Google Mountain View Wi-Fi network.

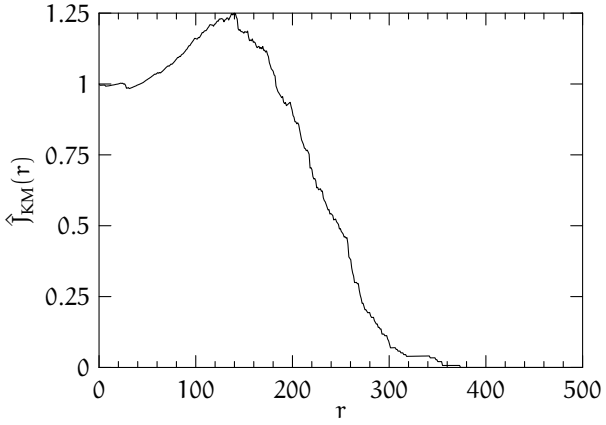

Fig. 2. The estimate of the J-function for the Google Mountain View data set.

Figure 1 shows the Wi-Fi access point data set, and Figure 2 gives the J-function estimate of the access point locations. From the J-function estimate we can clearly see that the data set is fairly regular at short distances, and heavily clustered at distance scales beyond few hundred meters. This is typical in planned deployments, since regularity arises naturally form coverage optimization and interference minimization, whereas large-scale population distribution induces the clustering effect. The plot also shows that approximation of the locations by a homogeneous Poisson point process would not be very appropriate at any length scale, so more elaborate models should be considered.

First step in the fitting process is to ensure that the assumptions made during the development of the theory are actually satisfied by the data set. In case we seek direct fit for the location distribution without considering covariates such as the

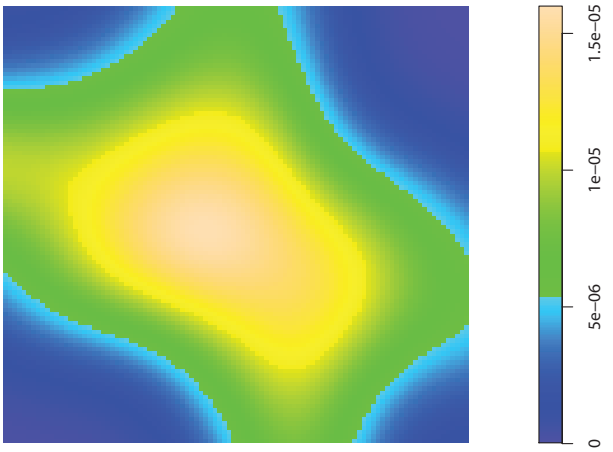

Fig. 3. The kernel density estimate of the intensity of the Mountain View data set.

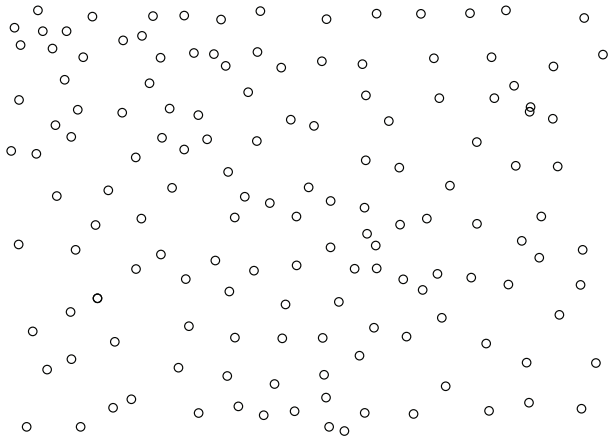

Fig. 4. The subset of the Mountain View data set used in fitting.

underlying population distribution, we should ensure that the data set considered is approximately stationary. We do this by studying the dependency of the intensity of the process by means of kernel density estimate, shown in Figure 4. The figure confirms the expectation that the overall density is much higher in the centre of the area, and thus there is a radial non-stationarity in the data. To compensate for this, we shall instead focus on a subset of data point shown in Figure 4, corresponding to area towards east from the centre in which the overall distribution is clearly close to stationary.

Now that the data set used for fitting is selected, we need to specify the model to be fitted. We shall apply the Geyer saturation process due to its versatility, and perform the fit by means of maximum pseudolikelihood as discussed above. Recall that the Geyer process has four parameters, the intensity parameter β, the interaction parameter γ, interaction distance r, and the saturation threshold ζ (the remaining parameter α is determined by these by normalization). First two of

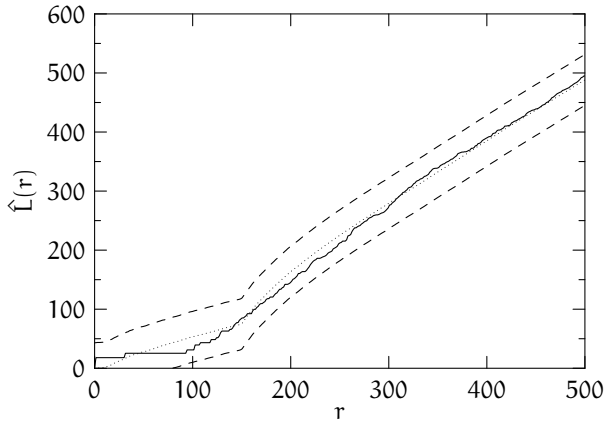

Fig. 5. The estimate of the L function for the data set together with the envelopes from 200 simulations of the Geyer saturation process with $\beta \approx 3.329 \times 10^{-5}$, $\gamma \approx 0.4112$, $\zeta = 2$ and $r = 150\,\mathrm{m}$.

these are regular parameters amenable to fitting, whereas the irregular parameters have to be inferred by other means. For estimating r, we use the J-function estimate which peaks at roughly $r = 150\,\mathrm{m}$. For the saturation threshold ζ lower values are preferred, since these make the evaluation of the log-pseudolikelihood computationally fast. Thus we first set $\zeta = 2$ and explore the results of the fit. Maximization of the log-pseudolikelihood resulted with these choices in the fits $\beta \approx 3.329 \times 10^{-5}$ and $\gamma \approx 0.4112$, consistent with a regular point pattern.

Next step is then to evaluate the goodness of the fit obtained. In general there are two approaches we can adopt. First is to study statistics of the original data set compared to the statistics of the obtained model. If the fit is good, key statistics such as the K-function should agree on both the data set and the model. Second approach is to study the residuals of the model as defined by equation (8). Figure 5 shows the L-function of the data set compared to the envelopes of the L-functions for 200 realisations of the fitted model obtained by means of Monte Carlo simulations. Numerical methods are required since no analytical expression for the envelopes is available for the Geyer saturation process. The L-function for the data set is clearly well contained within the numerically estimated envelopes for the model, indicating that the obtained fit is of high quality. We shall confirm this by studying the residuals of the log-pseudolikelihood. For models featuring interactions between points, such as the Geyer process, the appropriate diagnostic is the quantile-quantile plot of the residuals with envelopes again obtained from numerical simulations [22]. Figure 6 gives this plot for the data set and fit in question. Again, very good correspondence with the model and the data set is observed.

We shall conclude this section by briefly giving Geyer saturation model fits for other types of node location data sets as well, namely transmitter locations obtained from the licensing database of the Federal Communications Commission (FCC) of the USA. These data sets contain snapshots of the tower

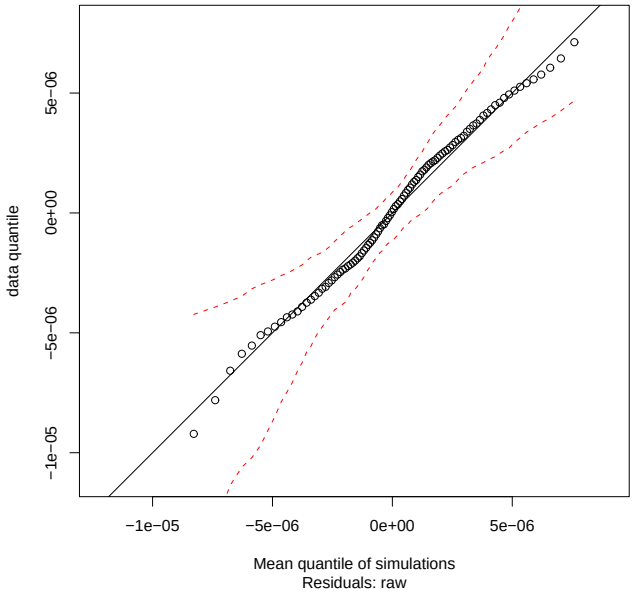

Fig. 6. The Q-Q plot of the Geyer fit for the Mountain View data.

TABLE I
THE FCC DATA SETS USED IN FITTING.

Data set	Number of points
AM stations	4934
FM stations	15535
NTSC TV stations	8922
Digital TV stations	1056
Cellular sites	21115

locations for a number of different transmitter technologies. We focus here on the five data sets given in Table I, consisting of AM and FM radio and TV transmitter locations, together with cellular radiotelephone service site locations.

We followed similar methodology for each data set as used above for the Mountain View case. First a region of stationarity was selected, in this case a rectangular region near the east coast. The irregular parameters used in the fitting were initially estimated from J-function and K-function plots, lowest reasonable value of ζ was used, and the maximum pseudolikelihood fit was obtained. The diagnostic plots, and especially the Q-Q plots were used to assess goodness-of-fit, and the need to change the chosen irregular parameter values. In each case very good fits were obtained both in terms of L-function estimates and the distribution of the residuals. The final values for the model parameters are summarized in Table II.

V. APPLICATIONS AND FUTURE WORK

The results given above show that the spatial structure of wireless communication networks in terms of node locations

TABLE II
PARAMETERS FOR BEST GEYER SATURATION MODEL FITS FOR THE FCC DATA SETS.

	Parameters		Fit	
Data set	r	ζ	β	γ
AM stations	10000	2	8.8400×10^{-10}	1.7268
FM stations	5000	4	1.5296×10^{-9}	1.6908
NTSC TV stations	10000	2	2.7496×10^{-10}	1.7995
Digital TV stations	50000	3	1.0528×10^{-10}	1.3435
Cellular sites	24000	2	1.6176×10^{-8}	0.5162

can be modeled effectively by the spatial statistics methods. We believe that such models provide very good compromise in terms of abstraction level between the overly simplistic Poisson modeling or direct application of techniques for modeling the connectivity of the network, and working with experimental data directly. The fitted models can reproduce key spatial statistics well for both planned and chaotic deployments, and are well suited for simulation work or semi-analytical approaches. The main alternative we plan to explore in future work is the use of optimization-based models as an alternative, potentially offering more explanatory power but having somewhat more limited applicability.

The key observation of the optimization-driven approach is that networks are usually deployed to serve a particular purpose, and are thus an outcome of a complex dynamic process. Successful fit of a point process model allows to replicate the outcomes of this process, and yields some information about statistics related to it, but does not in the end provide much insight to the internals of the process. Highly interesting alternative is to construct a concrete *deployment model* and study its outcomes using the techniques outlined here. Such a model would be simplest for tightly planned systems such as large-scale cellular networks, since the optimization problems the networks are deployed to solve are rather well known. This approach would closely parallel current developments in modeling the structure of fixed networks, as discussed in [1]. For chaotic deployments, such as placement of Wi-Fi access points owned by private citizens in urban areas, more complex models would appear to be necessary. Successful development of such models can shed light on the dynamics behind the actual network formation, which is one of the main challenges the emerging network science will have to deal with. Nevertheless, we see such optimization driven models as complementary to the spatial statistics based ones. While potentially offering more insight, they are tightly coupled on the "background", such as population distribution, the optimization problem takes as one of the inputs. Of course, it is also possibly to develop combined models, for example by modeling the background separately using the spatial statistics techniques, and then studying the outcome of the optimization process on top of the stochastic background model.

The developed node location models have a number of direct applications we are exploring at present. The most

straightforward one is to use them as parts of system level simulations or analytical calculations either as models of base station or access point locations, or those of user location distributions. Generation of realizations of these models is not expensive numerically, and implementations of the required algorithms are readily available. Much more interesting application is trying to use such models in run-time optimization tasks *within* a wireless network. This requires on-line estimation of model parameters or selection of a model from a collection of earlier fits (essentially resulting in a system identification problem), and then extracting the statistics of interest to the optimization task at hand. We are currently studying an application in *spectrum sensing*, related to dynamic spectrum access (DSA) networks. In particular, we apply numerical estimation of distance distributions of the fitted models in estimating detection and interference probabilities in such DSA scenarios. We believe that such on-line usages form a very promising application area for such stochastic models, and we are actively developing solutions for the related estimation problems.

VI. Conclusions

In this paper we studied the problem of modeling the spatial structure of wireless communication networks using a stochastic approach based on the theory of point processes and their spatial statistics. We showed that it is indeed possible to create *realistic* models of node locations with relatively simple techniques, and that a single parametric family of models, namely the Geyer saturation process, was capable of replicating the structure of several different network types. The resulting models can be used directly in simulation work or as basis for analytical calculations. We also discussed at length opportunities for future work, especially highlighting the need for both deepening the understanding of the processes by with networks are deployed, as well as the promising outlook on applications of such models in on-line network optimization problems. In general, we strongly argue that the spatial statistics approach provides a solid and general foundation for future work towards development of network science for systems in which the spatial component plays integral part.

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