Exploiting Spatial Statistics of Primary and Secondary Users towards Improved Cognitive Radio Networks

(Invited Paper)

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***Abstract*—We show how locations of primary and secondary users of spectrum can be characterized quantitatively using modern spatial statistics techniques. We argue that several interesting practical engineering applications can be based on these methods. In order to demonstrate the developed techniques, we will apply those to real-world data in particularly TV-transmitter locations in the USA. The spatial statistic techniques have a natural use in the analysis of Dynamic Spectrum Access (DSA) opportunities. We expect that the presented methods have strong potential to be used not only for theoretical work, but also as foundations for concrete protocols and algorithms for improved cognitive radios.**

I. Introduction

Cognitive Radios (CR) and Cognitive Wireless Networks (CWN) are promising new paradigms to apply machine learning and optimization methods to wireless communications [1]. Over the past years it has been recognized that the traditional approaches to spectrum management need to be challenged. This has lead to the concept of the first type of Cognitive Radio (CR-I), which could be described as spectrum agile and opportunistic radio. Generally speaking this Dynamic Spectrum Access (DSA) approach is currently a dominant subfield of cognitive radio research. The cognitive radios of the second kind (CR-II) follow the Mitola's original [1], much more challenging, vision on combining the machine learning and model based cognitive radios. CR-IIs are able to optimize their behavior based on environmental stimuli and context.

In this paper we focus mostly on DSA concepts in the context of using spatial statistics to understand the network topology of primary and secondary users. We argue that use of network topology information, including here the geometric relations of the nodes, can bring significant benefits to cognitive radios and networks. Although this invited paper is mainly focusing on introducing theoretical concepts of spatial statistics for CR, we are also demonstrating the potential of the approach with some real-world data. This work is a part of our work towards implementing topology awareness into the cognitive radio platforms as a topology engine module in a cognitive resource manager framework [2], [3].

II. Background on Dynamic Spectrum Access

A great deal of very promising research has been performed towards understanding fundamentals of Dynamic Spectrum Access and enabling technological solutions. It is beyond the scope of this article to provide a full review of this field. The reader should note that the research work in the field has gone beyond "simple" DSA work that has been considering open spectrum or dynamic spectrum auctions as necessary conditions for DSA solutions. The body of DSA work has become very sophisticated domain of research, which has applicability in many communications engineering sub-fields.

For general survey on spectrum sensing algorithms we refer to Yuek and Arslan [4]. Also the authors in [5], [6] provide excellent discussion on the coexistence bounds. There are number of papers discussing specific methods for DSA [7], [8]. Cooperation and learning in multiuser opportunistic DSA have been studied with partially observable markov decision process by Zhao *et al.* [9], [10]. An excellent and recent survey on DSA is provided by Zhao and Sadler in [11]. There has been less work in the domain of using location awareness for cognitive radios, the most notable and interesting exceptions being [12], [13]. The two cited papers have limited DSA context, but regardless they specifically consider location information. The radio environment maps (REM) concept for cognitive radios was introduced in [14]. This concept is closest to our approach, since it is specifically trying to describe the radio and interference environment in a parameterized form to local cognitive devices.

The issues of spatial statistics and topology has not been, in the best of our knowledge, discussed earlier in CR context except in [2]. The information on spatial statistics has a direct relevance both in practical design and theoretical analysis of DSA scenarios. The spatial distribution of primary and secondary users, and cross-correlations between them, naturally limits DSA opportunities and provide theoretical foundations to statistically estimate for example false-detection limits. In the practical domain, we have been lately studying issues such as how much information can be saved on exchanging only statistical information of pair-correlations between radio positions instead of reporting complete location information for all terminals.

III. Characterising Locations of Primary Users

We shall now present and study a number of methods for characterizing and reasoning about location distributions of primary and secondary users. In order to do this we need to introduce some fundamentals of spatial statistics, namely the theory of *point processes*. Since we are targeting for a very application-driven viewpoint, for our purposes it suffices to think of point processes as random collections $X = \{x_i\}$ of points $x_i \in U$ on some region $U \subset \mathbb{R}^n$ (in addition to the locations, the number of points might be random as well). We consider the locations of primary and secondary users to be such realizations, with the underlying point process being unknown[1]. The mathematically advanced reader can find more rigorous definitions in terms of sequence-valued maps or random counting measures from, for example, [15], [16]. We shall also restrict our discussion to the two-dimensional case for simplicity, which is sufficient as long as the curvature of Earth and elevation of primary and secondary users can be ignored. Nevertheless, the generalisation to the three-dimensional case is not difficult.

Fig. 1. Excerpt from the FCC database on NTSC television station transmitter sites.

Throughout this section we shall use the data set depicted in Fig. 1 as our case study. It consists of the locations of 8922 NTSC television station transmitter sites obtained from the FCC database. We chose this particular data set due to its relevance for DSA applications, tv-bands often being cited as prime candidates for spectrum reuse. We shall use this data set to introduce by means of examples various statistics point processes can be characterized with, followed in the next section by a more detailed look at fitting synthetic models to the data for, e.g., reasoning and simulation purposes.

We begin by looking at distance-based statistics. Typically we either consider the distribution of the distance from a randomly selected point of U into the nearest point of X, or the distribution of the distance from a randomly selected point of X to its nearest neighbor. The former yields the *spherical contact distribution function* $H_s(r)$, whereas the latter gives the *nearest-neighbour distance distribution function* $D(r)$. Being distribution functions, both of these vanish at origin and approach unity as r becomes large. Estimates of $H_s(r)$ and $D(r)$ for the FCC dataset are given in Fig. 2 and Fig. 3, respectively. By comparing the two curves we immediatelly see that $D(r)$ grows much faster for small values of r. This is an early indication of *clustering* in the data. Another way to make this deduction is to look at the so-called J-statistic [17], defined by

$$J(r) \equiv \frac{1 - D(r)}{1 - H_s(r)}. \tag{1}$$

For uniformly distributed points (defining the homogeneous Poisson point process assuming the total number of points is Poisson) $D(r)$ coincides with $H_s(r)$, so $J(r) \equiv 1$. For regular processes (such as lattices) we would expect $J > 1$, whereas for clustered processes we should have $J < 1$. The estimate for the J-statistic of our data set is shown in Fig. 4. The rapid departure from unity towards zero further highlighting the clustered nature of the transmission site locations.

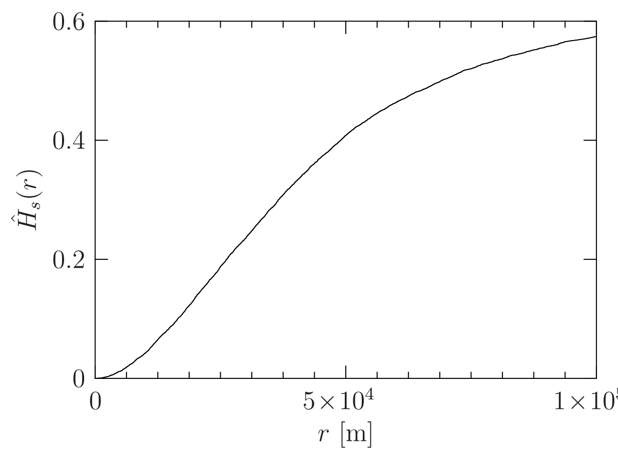

Fig. 2. Estimate of the spherical contact distribution function for the data set of NTSC television station transmitter sites.

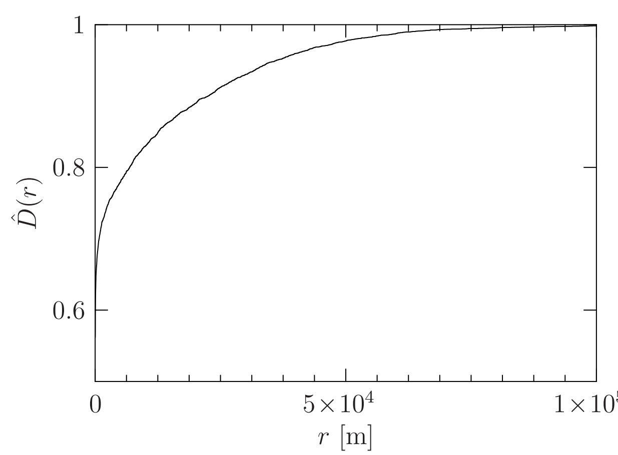

Fig. 3. Estimate of the nearest-neighbour distance distribution function for the data set of NTSC television station transmitter sites.

[1]Notice that this is simply the generalisation of the approach adopted when applying the theory of time series into the spatial case. Time series are special cases of point processes on open subsets of the real line.

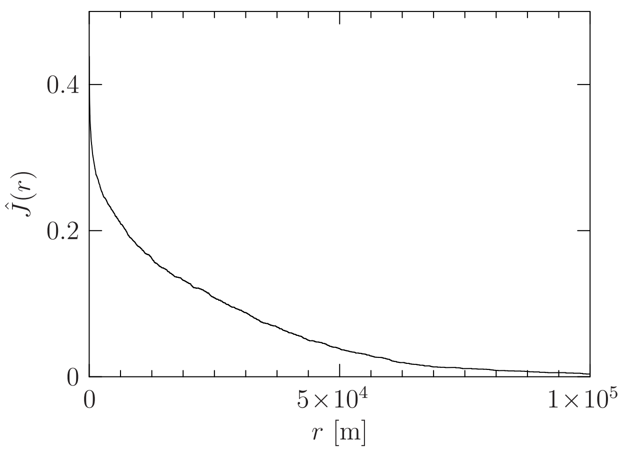

Fig. 4. Estimate of the J-statistic for the data set of NTSC television station transmitter sites.

While the distance-based statistics introduced above are certainly expressive, their use is not without problems. They are not as robust as second-order statistics (but this should be expected from the parallel to basic queuing theory) and their power in modelling applications is limited. Because of this, we shall briefly introduce the basic second- and higher-order statistics that can be used to describe point patterns. Perhaps the most fundamental second-order statistic is the *pair-correlation function* $\xi(r)$ defined by the joint probability density

$$dP = \nu^2\big(1 + \xi(r)\big)\, dA_1\, dA_2 \tag{2}$$

of finding one point in each of the two area elements dA_1 and dA_2 (ν is the *intensity* of the process, giving the mean number of points per unit area). Clearly if $\xi \equiv 0$ no obvious correlation is present as one would expect for, say, the homogeneous Poisson point process. The definition of the pair-correlation function can, of course, be extended to the case of n-point correlations for any $n > 1$. The definition of these in a manner analogous to (2) is given by the joint probability density

$$dP = \nu^n\big(1 + \xi^{(n)}(r_{1,2}, r_{1,3}, \ldots, r_{n-1,n})\big)\, dA_1 \cdots dA_n, \tag{3}$$

of finding a point in each of the area elements dA_i, where $r_{i,j}$ denotes the distance between area elements dA_i and dA_j. For example, considering correlations up to the three-point case yields

$$\begin{aligned} dP &= \nu^3\big(1 + \xi^{(3)}(r_{1,2}, r_{1,3}, r_{2,3})\big)\, dA_1\, dA_2,\, dA_3 \\ &= \nu^3\big(1 + \xi(r_{1,2}) + \xi(r_{1,3}) + \xi(r_{2,3}) \\ &\qquad + \zeta(r_{1,2}, r_{1,3}, r_{2,3})\big)\, dA_1\, dA_2,\, dA_3, \end{aligned} \tag{4}$$

where ζ is the *reduced three-point correlation function*, expressing the residual three-point correlations that do not arise from the two-point contributions directly. For example applications of pair correlation functions in studying wireless networks in general and for the accompanying discussion, see [18], [2].

Another commonly used second-order statistic related to the pair-correlation function is the Ripley's K-function, defined in general by the equality

$$\xi(r) \equiv \frac{1}{d b_d r^{d-1}} \frac{dK(r)}{dr} - 1, \tag{5}$$

where b_d is the volume of the d-dimensional unit ball. This normalization is chosen to have $K(r) = \pi r^2$ for the Poisson case in the plane (and analogously for higher dimensions). Fig. 5 shows the estimate of the K-function of our data set compared to the Poisson expectation, cementing our conclusion on the clustered structure of the data.

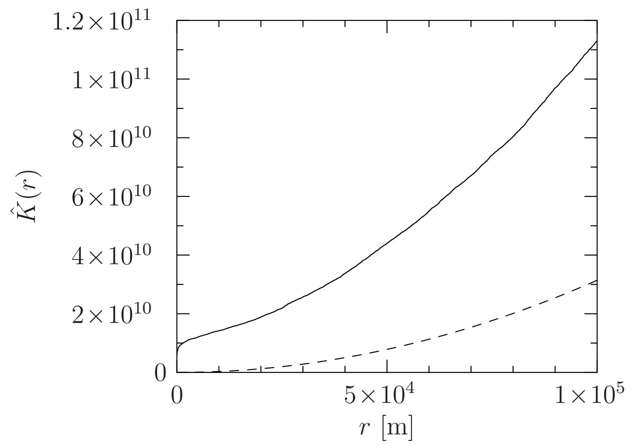

Fig. 5. Estimate of the Ripley K-function for the data set of NTSC television station transmitter sites. Dashed line shows theoretical value for the case of complete spatial independence (Poisson case).

IV. Constructing Models from Experimental Data

We shall now move on from analyzing and characterzing point distributions into modeling them. Our approach will be mainly parametric, that is, we shall study the problem of determining the parameters of an analytically solvable point process model so as to provide an "optimal" fit to the data in the sense of a metric of interest. Such models can then be applied either in analytical work or as parts of simulations to give an improved estimates of location distributions of primary and secondary users compared to the typically assumed homogeneous Poisson case, which we have already seen to be unrealistic.

We begin by considering techniques motivated by Bayesian reasoning. The most common technique applied in the spatial statistics literature concerning numerical work is the method of maximum pseudolikelihood [19]. A practical version of the algorithm suitable for point process applications has been given by Baddeley and Turner [20], who have also given an implementation publicly available as a part of their spatstat-package [21] for the R environment [22]. A more statistically satisfying alternative is the approximate likelihood method of Huang and Ogata [23], but this is significantly more expensive computationally. In our work we have found the pseudolikelihood to be quite sufficient. As an example application we consider creating a model for the east coast part of our tv-station data set shown in the left panel of Fig. 6. Looking

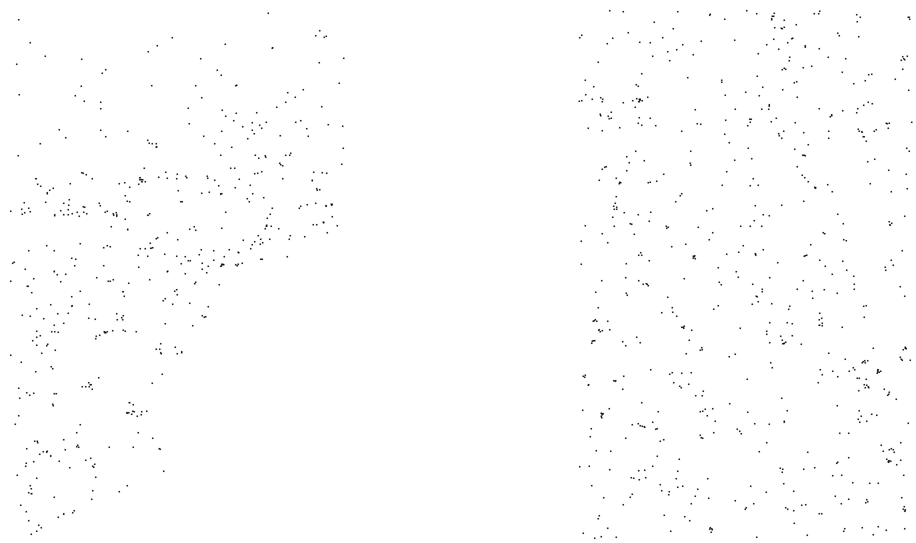

Fig. 6. An excerpt from the tv-station data corresponding to the east coast together with a realisation of a model fitted with maximum pseudolikelihood technique.

at the J- and K-statistics confirms that also this excerpt is strongly clustered, so the model to be fitted should be a clustered one as well. We chose after some experiments the Geyer saturation process [24] because of its ability to model both regular and clustered point patterns. The model is parameterised by an overall intensity β, interaction radius r and an interaction parameter γ with $\gamma < 1$ corresponding to regular case and $\gamma > 1$ indicating clustering. We obtained a good fit after choosing interaction radius of 10000 (estimated from the J-statistic), resulting in maximum pseudolikelihood estimates $\hat{\beta} \approx 2.7496 \times 10^{-10}$ and $\hat{\gamma} \approx 1.7995$. A realisation of the model obtained by means of the Metropolis-Hastings algorithm is depicted in the right panel of Fig. 6. The similarity in structure is evident.

The second fitting approach we shall consider is the *method of minimum contrast* of Diggle and Gratton [25]. In this approach one selects a suitable summary statistic S, such as the K-function or (equivalently) the pair correlation function, and a point process model with adjustable parameters α_i. Ideally the model such be chosen to yield analytically tractable, explicit expressions $S(\alpha_i)$ for the summary statistic as the function of the adjustable parameters. One then calculates an estimate $\hat{S}$ from the data, and determines the values of α_i minimizing the difference of $\hat{S}$ and $S(\alpha_i)$ in some reasonable sense. Waagepetersen has proposed in [26] a criterion of the form

$$C(\alpha_i) = \int_{r_0}^{r_1} \left(\sqrt[4]{\hat{S}(r)} - \sqrt[4]{S(\alpha_i; r)} \right)^2 dr. \qquad (6)$$

While the minimum contrast method is certainly interesting, we have found it really useful only in limited cases. Usually one encounters either the problem of the summary statistic being without analytical form, or the underdetermination problem of the optimization of $C(\alpha_i)$ ending up on a pareto frontier instead of a single point. There are, however, certain special cases of the minimum contrast method that can be very powerful, and these are the ones we shall focus on now.

A highly interesting variation of the minimum contrast principle can be obtained by using the pair-correlation functions as the summary statistic, and choosing the model to be a variation of the Gauss-Poisson process proposed by Kerscher [27]. The original Gauss-Poisson process is defined as follows. First, one generates a sample of a Poisson process of some intensity ν. Then one associates to each point a cluster of zero, one or two points with (fixed) probabilities p_1, p_2 and p_3 respectively. In the single point case the cluster point is located at the cluster centre. In the case of two-point cluster, the points are typically taken to be at a unit distance apart, with the cluster centre at the midpoint between the cluster points. However, for our applications (see below) it is better to let the inter-point distance be a random variable, preferably with a density $f(r)$.

What Kerscher observed is that the Gauss-Poisson process is the point process analogue of the normal distribution, completely defined by its first- and second-order characteristics. This approach also suggests that suitable Gauss-Poisson process can be used to generate point distributions with almost arbitrary two-point correlation function $\xi(r)$. As Kerscher showed, this is indeed the case. One simply takes $p_1 = 0$, makes $f(r)$ proportional to the desired two-point correlation function and chooses the constant of proportionality C, ν and p_2 by $C^{-1} = \int_E \xi(|x|)\, dx$, $\nu = \nu_{\mathrm{GP}}\big(1 - \nu_{\mathrm{GP}}/(2C)\big)$ and $p_2 = 1 - \nu_{\mathrm{GP}}/(2C - \nu_{\mathrm{GP}})$ to obtain the desired intensity ν_{GP} for the outcome. Finally, Kerscher has also shown that the same reasoning can be applied to obtain natural generalisations of the Gauss-Poisson process specified by the subsequent n-point correlations. Thus, to generate models replicating the observed n-point correlation structures in wireless networks (such as the ones reported in [18]), these extensions of the Gauss-Poisson process offer exactly the right amount of flexibility.

We conclude this section by noting that it is also possible to generate models driven by some other spatial phenomenon of which data is available. For example, it is expected that for wireless technologies with shorter transmission ranges transceiver locations closely follow population densities[2]. Accordingly, measured population densities $\lambda(x)$ can be used as the intensity functions of, for example, a (inhomogeneous) Poisson process, yielding an implementation-wise simple stochastic location model. Figure 7 shows as example realisation of such a process driven by population data of Belgium obtained from the LandScan dataset [28].

Fig. 7. A point process model driven by the population data of Belgium.

V. Applications of PU Distributions

We shall consider two further applications of primary user location data, whether obtained from measurements or as a realisation of a model constructed using the above principles. In first of these applications we use the PU location data to create models of client locations. For example, in case of the tv-station location data we can create some rough models on the distribution of receivers as well. As the second application we consider studying of the structure of service areas of the primary users. We use spatial tessellations, subdivisions of our domain U into regions, as the foundation of both applications. In particular we consider *Voronoi diagrams* [29], [30]. Recall that the *Voronoi polygon* of $x_i \in X$ is the closed set

$$V(x_i) \equiv \left\{ y \in \mathbb{R}^2 \,\middle|\, \|y - x_i\| \le \|y - x_j\|, j \neq i \right\}, \qquad (7)$$

and the collection of all the Voronoi polygons is the *Voronoi diagram* generated by the point process X.

We first use the Voronoi diagram $\mathcal{V}(X)$ to define a family of simple population or user models. Namely, we define a population density function in a per-cell fashion by putting

$$\rho(x) \equiv \nu_0 \sum_i [x \in V(x_i)] w(|V(x_i)|), \qquad (8)$$

where $[\cdot]$ is the Iverson bracket (with value of one if the enclosed proposition is true, and zero otherwise), $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a weight function and $|V(x_i)|$ is the area of the corresponding Voronoi polygon. We can now use ρ as the intensity of another point process, and consider the realizations of that process as examples of random user populations (see Fig. 8 for an illustration). This model can also be made analytically tractable by considering sufficiently simple processes as forming the "driving" point distribution X. Large-scale inhomogeinities can also be modelled while retaining the Poisson character by modulating the original process driving the Voronoi diagram by, for example, a Boolean process. See [31] for an example of such a modulation of the Voronoi diagram.

[2]There probably is a further effect due to income differences, but this is harder to quantify and we shall not focus on that here.

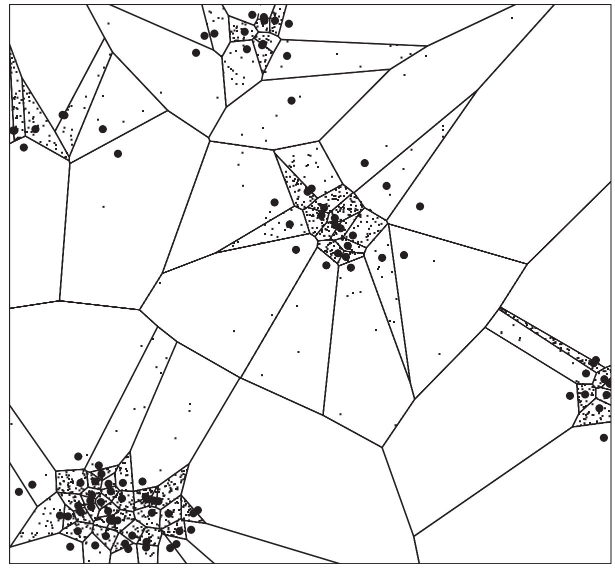

Fig. 8. A realisation of a Cox process driven by areas of Poisson-Voronoi regions.

As the second application we now consider the rough characterisation of the service areas for different types of point distributions. We continue to model these service areas as Voronoi polygons. Despite the relative simplicity of this approach, this is not really a very drastic approximation. For example, if all the points would have transmitters of equal power and propagation model is isotropic, tuning in to the strongest signal results precisely the service areas described by the Voronoi diagram. We can now characterise the Voronoi polygons in several ways. The statistics of interest especially in the presence of client mobility are typically the distribution of the areas of the polygons, the number of polygon edges and the shapes of the individual regions. For simplicity, we consider here only *normalized areas*, that is, we choose our units to yield $\mathbb{E}\{|V(x_i)|\} = 1$. As the baseline we consider Voronoi diagrams generated by homogeneous Poisson processes (the *Poisson-Voronoi diagrams*), as their statistics are very well known (see, for example, [32], [33]).

The distribution of the cell areas for the tv-station data as well as the Poisson-Voronoi case are depicted in Fig. 9. We see very clearly that the area distribution for the former case has a substantially heavier tail, while having a considerable peak at small cell sizes. This kind of behaviour is typical in Voronoi diagrams generated by clustered point patterns. The significant departure from Poisson case further highlights the need to

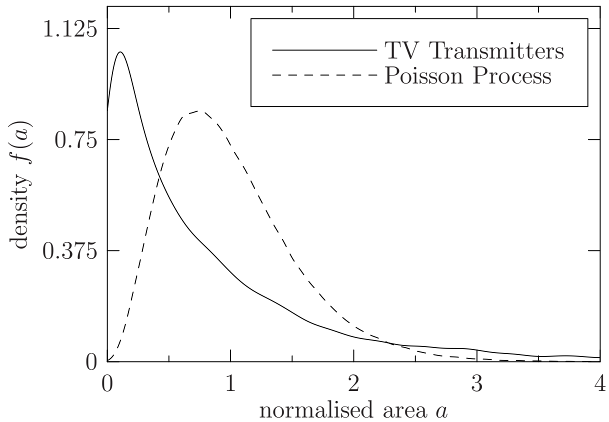

Fig. 9. Distribution of normalized areas of Voronoi polygons for the Poisson-Voronoi and tv-station cases.

consider a wider spectrum of models of location distributions both in analytical work and in simulations. This comparison can obviously be extended to cover other quantities besides the areas of the cells. For example, letting A denote the area of a cell in the normalizing units, and P similarly normalized perimeter, we could characterize the shapes of the polygons by their *form factor* $F \equiv 4\pi A/P^2$. The form factor characterises the "roundness" of the cell, with the limits $F \to 1$ as the cell approaches a circular disc, and $F \to 0$ as the cell becomes more and more elongated. Our initial analysis indicates that the departure from the Poisson-Voronoi case is quite evident in this statistic as well.

VI. Conclusions

We have shown how locations of primary and secondary users of spectrum can be characterized quantitatively using modern spatial statistics techniques. We foresee several interesting applications for these techniques that ought to be studied in the future. Development of practical models for simulation and analysis purposes based on empirical data is certainly an important step in increasing our confidence on performance evaluation carried out on any proposed dynamic spectrum access scheme. We also expect that we can go further than that, and actually use spatial statistics techniques as a foundation of new types of spectrum sharing algorithms and protocols. This brings on a complicated set of problems ranging from on-line estimation of geometry information to effectively sharing and applying it. These are examples of the issues we intend to address in our future work in this field.

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