This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the ICC 2008 proceedings.

Throughput Analysis of Frequency-Agile Medium Access Control Protocols

Alexandre de Baynast, Lili Wu and Petri Mähönen
Department of Wireless Networks
RWTH Aachen University
Kackertstraße 9, D-52072 Aachen, Germany
Email: ade@mobnets.rwth-aachen.de

***Abstract*—In this paper, we analyze three opportunistic multi-channel MAC protocols that associate a newcoming terminal station to one of the access points available within its transmission range: 1) by selecting the access point with best signal-to-noise-ratio transmission link, the protocol "max SNR" achieves selection diversity gain, 2) by selecting the access point with lowest load, the protocol "min load" achieves "load balancing" gain, 3) by selecting the access point which provides the best throughput among all access points, the protocol "max throughput" achieves the optimum trade-off between selection diversity gain and occupancy. All three protocols achieve significant gain against the random association scheme in an opportunistic and distributed manner. Our calculations based on the bins and balls analysis shows that protocols "max SNR" and "min load" are near-optimal at low SNR and high SNR, respectively.**

I. Introduction

In wireless networks, signal fading is a particularly severe channel impairment that can be mitigated through the use of diversity. Space or multiple-antenna diversity techniques are particularly attractive as they can be readily combined with other forms of diversity, e.g., time and frequency diversity, and still offer dramatic performance gains when other forms of diversity are unavailable. In contrast to the more conventional forms of space diversity with physical arrays [1], this work examines the problem of exploiting space diversity for association between a newcoming Terminal Station (TS) and several available Access Points (APs) within its transmission range. In order to design a Medium Access Control (MAC) protocol which fully benefits from this spatial diversity, MAC layer parameters have also to be considered. In particular, the load of each AP has to be taken into account during the association process. Along the lines of [2], building on the IEEE 802.11 standard [3], we analyze three opportunistic multi-channel MAC (OMC-MAC) protocols for AP association: 1) by selecting the AP with best signal-to-noise-ratio (SNR) transmission link, the protocol "max SNR" achieves selection diversity gain, 2) by selecting the AP with the smallest number of active users (lowest load), the protocol "min load" achieves selection occupancy gain, 3) by selecting the AP which provides the best throughput among all APs, the protocol "max-throughput" achieves the optimum trade-off in term of throughput between the transmission diversity gain and occupancy. All three OMC-MAC protocols have the following key features: 1) they achieve selection gain in an opportunistic and distributed manner; 2) the size of the contention window is adjusted adaptively based on the load of the associated AP; and 3) these protocols achieve "resource-pooling" and thus improve the stability of the network. Whereas we focus on IEEE 802.11 standard-based transmission system, our analysis can be readily extended to any orthogonal multiple access system.

II. System Model

In our model for the wireless channel in Fig. 1, narrow-band transmissions between the TSs and APs suffer the effects of frequency nonselective fading and additive noise. Our analysis in Section III considers the case of slow fading to capture scenarios in which the channel coherence time is typically of the order of a few tens or hundreds of symbol periods, and measures performance by average throughput. Specifically, we assume throughout the paper that the channel fading coefficients remain constant during a physical layer packet duration (typically a few hundreds of bits) but vary from one MAC packet to another, whose duration typically is up to one order of magnitude longer than for a physical layer packet [4], [5]. While these protocols can be naturally extended to the kind of highly mobile scenarios in which frequency- and time-selective fading, respectively, are encountered, their potential impact becomes less substantial when other forms of diversity can be exploited in the system.

A. Medium Access

As in many current wireless networks, such as cellular and wireless LANs, we assume that the APs within the transmission range of TS of interest use orthogonal channels such that they do not interfere with each other. TSs communicate with APs using CSMA/CA protocol as in IEEE 802.11 standard [3]. Under this assumption, no packet/sample synchronization between TSs and APs is expected that greatly simplifies the implementation of the communication protocols. At each AP, collisions may occur between TSs connected to this AP. In order to avoid collisions, CSMA/CA protocols typically adopt an exponential backoff scheme, in which a contention window initiated with a minimum size can be adapted exponentially up to a maximum size in case of collision. If all TSs are fully backlogged, the optimal transmission probability of the CSMA/CA system is roughly equivalent to a TDMA system

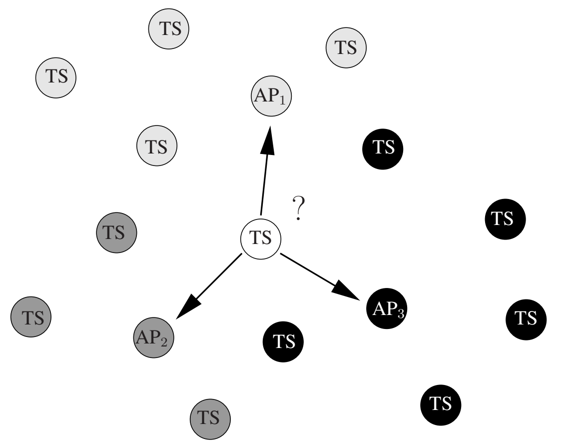

Fig. 1. System model: A newcoming terminal station seeks to establish connection with one of the access points AP_1, AP_2 and AP_3 that are within its transmission range. Each transmission link from TS to AP_i, $i \in \{1, 2, 3\}$ is modeled as additive white Gaussian noise with flat fading coefficient a_i. In this example, the current load is 4 for AP_1, 3 for AP_2 and 5 for AP_3.

with successful transmission time $1/\alpha k$, where α (equivalent to the coefficient K in [6, Eq. (29)]) is a constant depending on the transmission parameters at the physical and MAC layer and k is the current load of the considered AP (excluding TS of interest).

B. Equivalent Propagation Channel Models

Under the above orthogonality constraints, we can now conveniently, and without loss of generality, characterize our channel models using a time-division notation. Specifically, we model the transmission between TS and $AP_i, i = 1, \ldots, N$, as

$$y_i[n] = a_i[n]x[n] + z_i[n], \; n = 1, \ldots, \infty \tag{1}$$

where $x[n]$ is the transmitted signal from TS and $y_i[n]$ is the destination received signal at AP_i. In (1), $a_i[n]$ captures the effects of path-loss, shadowing, and frequency nonselective fading, and $z_i[n]$ captures the effects of receiver noise and other forms of interference in the system. We consider the scenario in which all fading coefficients $a_i[n]$ and the loads are known or accurately measured by the TS of interest. This estimation can be efficiently carried out at the beginning or at the end of each physical layer packet transmission, based on the preamble. Then, the estimates of the coefficients $a_i[n]$ are fed back to TS through a feedback channel. We model $a_i[n]$ as zero-mean, independent, circularly symmetric complex Gaussian random variables with unit variance.

C. Parameterizations

Two important parameters of the system are the average SNR and the normalized spectral efficiency with a single AP. Assuming that the noise is modeled as zero-mean complex Gaussian random sequences with variance σ^2, the SNR can be represented as

$$\mathrm{SNR} = \frac{P}{\sigma^2}, \tag{2}$$

where P is the transmission power. In addition to SNR, transmission systems are further parameterized by the normalized rate C_{norm}, or spectral efficiency attempted by the transmitting TS. Nominally, we parameterize the system by the pair (SNR,C_{norm}) where

$$C_{\text{norm}} = \frac{C}{\langle C(\beta+1, \mathrm{SNR})\rangle}. \tag{3}$$

Here $\langle C(\beta+1, \mathrm{SNR})\rangle$ denotes the CSMA/CA-based multiple access channel capacity for Rayleigh fading environment for a single AP with current load β (without the newcoming terminal station) [7]:

$$\langle C(\beta{+}1, \mathrm{SNR})\rangle = \frac{-1}{\alpha(\beta+1)\log 2}\exp(-1/\mathrm{SNR})\mathrm{Ei}\left(-1/\mathrm{SNR}\right) \tag{4}$$

where $\mathrm{Ei}(x)$ is the exponential-integral function [7, Eqs. (4-7)]. The coefficient α, as shown in [6], depends only on the transmission parameters at the physical layer and the access scheme adopted, and is independent of the current load β. The term $1/\alpha(\beta+1)$ represents the optimal probability of transmission for each TS that maximizes the throughput of the CSMA/CA system [6, Eq. (28)]. The quantity $-\exp(-1/\mathrm{SNR})\mathrm{Ei}\left(-1/\mathrm{SNR}\right)/\log 2$ denotes the channel capacity for a single user in Rayleigh fading environment (in bits per second per Hertz) [7, Eq. (5)]. Therefore, C_{norm} in (3) is the spectral efficiency normalized by the maximum achievable spectral efficiency for single AP and current load β.

III. Summary of the Results

Denote L as the number of active users that are randomly associated to N APs within the transmission range of the TS of interest. The key ratio between the total number of users L and the number of APs, i.e., the average load per AP, is denoted by

$$\beta = \frac{L}{N}. \tag{5}$$

A. Random Association

Assume that $L = \beta N$ terminals are *uniformly* randomly distributed among N APs. In that case, the load over APs follows a binomial distribution of parameter β. Assuming random association between newcoming TS and AP_i, $i = 1, \ldots, N$ and CSMA/CA MAC protocol as discussed above, the normalized achievable rate of this link $C_{\text{norm}}^{\text{rand}}$, achieved by independent and identically distributed zero-mean, circularly symmetric complex Gaussian inputs, is given for large N (and large $L = \beta N$) by:

$$C_{\text{norm}}^{\text{rand}} = \left(1 + \frac{1}{\beta}\right) \cdot \left(1 - \frac{1}{\exp(\beta)}\right) + \mathcal{O}\left(\frac{1}{\log^{\frac{3}{2}}(N)}\right). \tag{6}$$

The proof as the other proofs of this section can be found in [8]. Interestingly, the average rate (6) is always larger than the rate with single AP and load β. For large load β and large N, the gain vanishes to one, i.e., a system with single AP and a system with multiple APs based on random access scheme and same average load β, achieve the same spectral efficiency.

In addition to the average achievable rate, it is also interesting to determine a lower bound for the spectral efficiency which corresponds to the worst case in term of load, i.e., when the association is done with AP with largest load. In the case of a binomial load distribution with N large, the average throughput achieved with the described frequency agile MAC protocol, is equal to or larger than, with high probability,

$$C_{\text{norm}}^{\text{rand, worst case}} \geq \begin{cases} \frac{\beta+1}{1+2\beta e+2\log N}, & 1<\beta<\log N \\ \frac{\beta+1}{1+\beta+\sqrt{8\beta\log N}}, & \beta \geq \log N \end{cases} \quad (7)$$

which is significantly smaller than the worst case achievable rate obtained by Berenbrink's balanced loading algorithm [9], $C_{\text{norm}}^{\text{rand, worst case in [9]}} = 1/[1+(\log\log N)/(\beta+1)\log 2]$.

B. Association based on maximum SNR criterion

Assuming CSMA/CA protocol and random association between newcoming TS and AP_i, $i \in \{1,\dots,N\}$ such that the channel SNR between TS and AP_i is maximal over all N APs, the achievable rate of this link is given for large N by:

$$C_{\text{norm}}^{\text{max SNR}} = \begin{cases} (\gamma+\log N)\cdot\left(1+\frac{1}{\beta}\right)\cdot\left(1-\frac{1}{\exp(\beta)}\right)+\mathcal{O}\left(\frac{\text{SNR}^2}{\log^{\frac{1}{2}}(N)}\right) & \text{at low SNR} \\ \left(1+\frac{\log(\gamma+\log N)}{\log\text{SNR}-\gamma}\right)\cdot\left(1+\frac{1}{\beta}\right)\cdot\left(1-\frac{1}{\exp(\beta)}\right) +\mathcal{O}\left(\frac{\log(\text{SNR})}{\text{SNR}\log^{\frac{3}{2}}(N)}\right) & \text{at high SNR.} \end{cases} \quad (8)$$

The proof for the low SNR case can be found in [2, Eq. (7)], where the sum $\sum_{i=1}^{N} 1/i$ is approximated by $\log(N)+\gamma$ with N large. At high SNR, random association and max SNR-based association schemes achieve similar throughput. It was expected since at high SNR, any TS-AP link guarantees reliable transmission so the selection is less useful. Nevertheless, this protocol leads to SNR gain of at least $\log(\gamma+\log N)$ which may be not small for large number of APs and this, at any SNR. As for the random association scheme, it is also interesting to evaluate the achievable rate in the worst scenario. Since the load distribution is the same as for the random association scheme, we can directly combine results from Section III-A with (8), i.e.,

$$C_{\text{norm}}^{\text{max SNR, worst case}} \geq \begin{cases} \frac{(\beta+1)(\gamma+\log N)}{1+2\beta e+2\log N}, & 1<\beta<\log N, \text{ low SNR} \\ \frac{(\beta+1)(\gamma+\log N)}{1+\beta+\sqrt{8\beta\log N}}, & \beta\geq\log N, \text{ low SNR} \\ \left(1+\frac{\log(\gamma+\log N)}{\log\text{SNR}-\gamma}\right)\cdot\frac{\beta+1}{1+2\beta e+2\log N}, & 1<\beta<\log N, \text{ high SNR} \\ \left(1+\frac{\log(\gamma+\log N)}{\log\text{SNR}-\gamma}\right)\cdot\frac{\beta+1}{1+\beta+\sqrt{8\beta\log N}}, & \beta\geq\log N, \text{ high SNR.} \end{cases} \quad (9)$$

Contrary to rate (7) in the case of random association, the achievable rate in the worst case does not vanish for large N at low SNR. Typically, the achievable rate order for large N is $\mathcal{O}(\beta)$ at low SNR and $\mathcal{O}(1)$ at high SNR.

C. Association based on minimum load criterion

Assuming CSMA/CA protocol and association between newcoming TS and AP_i, $i \in \{1,\dots,N\}$ such that AP_i has the lowest load over all N APs (if several APs have the same minimum load, pick one randomly among those), the normalized achievable rate of this link, achieved by independent and identically distributed zero-mean, circularly symmetric complex Gaussian inputs, is given for large N by

$$C_{\text{norm}}^{\text{min load}} = (\beta+1)\left\{\sum_{k=0}^{\lfloor\beta\rfloor}\frac{1}{k+1}\sum_{j=1}^{N}(-1)^{j+1}\binom{N}{j}\times \left[\binom{\beta N}{k}\cdot\frac{1}{N^k}\left(1-\frac{1}{N}\right)^{\beta N-k}\right]^j\times \left[\sum_{l=k}^{\beta N}\binom{\beta N}{l}\cdot\frac{1}{N^l}\left(1-\frac{1}{N}\right)^{\beta N-l}\right]^{N-j}\right\}. \quad (10)$$

Since the minimum load over all APs is at most $\lfloor\beta\rfloor$ with probability one, the normalized achievable rate for the worst scenario in term of load is always equal to or greater than one and can be expressed as

$$C_{\text{norm}}^{\text{min load, worst case}} \geq \frac{1+\beta}{1+\lfloor\beta\rfloor}. \quad (11)$$

D. Association based on throughput maximization

Assuming MAC CSMA/CA protocol and association between the TS of interest and access point AP_i, $i\in\{1,\dots,N\}$ whose link has the highest throughput over all N APs, the normalized achievable rate, achieved by independent and identically distributed zero-mean, circularly symmetric complex Gaussian inputs, is by definition lower bounded for large N,β as:

$$C_{\text{norm}}^{\text{max throughput}} \geq \max\left\{C_{\text{norm}}^{\text{max SNR}}, C_{\text{norm}}^{\text{min load}}\right\}. \quad (12)$$

Whereas the protocol "max throughput" achieves the largest average throughput, from the implementation point of view, the computational complexity for determining the best AP to associate with, is slightly higher than for the other two protocols. Indeed, it requires a comparison between all terms $\log(1+|a_i|^2\text{SNR})/\tau_i,\ i=1,\dots,N$ instead of comparisons between all $|a_i|^2$ for the "max SNR" protocol or the loads of all APs for the "min load" protocol. However, as long as the number of APs is not very large, the computational complexity in all three association schemes remains rather small.

IV. Discussion

In this section, we compare the spectral efficiency results developed in Section III in various regimes. We divide the exposition into two sections. Section IV-A considers Gaussian inputs. The primary observations of this section are comparing the performance of the association protocols and examining how average load β and the number of APs affect that comparison. Section IV-B focuses on results for OFDM-based simulations that follow the IEEE 802.11g standard.

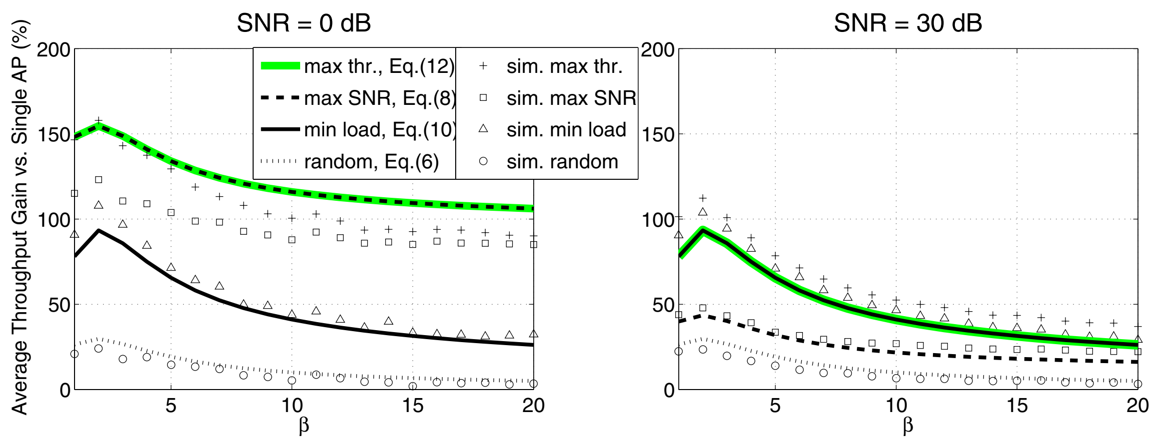

Fig. 2. Normalized average throughput $C_{\text{norm}}^{\text{rand}}$, $C_{\text{norm}}^{\text{max SNR}}$, $C_{\text{norm}}^{\text{min load}}$ and $C_{\text{norm}}^{\text{max throughput}}$ given by (6), (8) and (10) and (12), respectively as a function of the average load β per access point AP for $N = 4$ access points within the transmission range of the terminal. In this example, low and high SNR regime are considered: SNR= 10 dB (left side) and SNR= 30 dB (right side). The Matlab-based simulated throughput gains for all four protocols are also plotted.

A. Gaussian inputs

In Sections III-A—III-D, we have assumed that the TSs are uniformly randomly distributed among the APs. Then, the load distribution of the APs is binomial. In Fig. 2, we compare normalized throughput $C_{\text{norm}}^{\text{rand}}$, $C_{\text{norm}}^{\text{max SNR}}$, $C_{\text{norm}}^{\text{min load}}$ and $C_{\text{norm}}^{\text{max throughput}}$ given by (6), (8) and (10) and (12), respectively as a function of the average load β per access point AP for $N = 4$ APs within the transmission range of the terminal of interest. We have assumed that all APs are transmitting on mutually orthogonal channels without interfering with each other and each packet is experiencing Rayleigh fading during the transmission. Two cases are presented: the low SNR regime with average SNR= 0 decibel and the high SNR regime with SNR= 30 decibels. At low SNR, it is more important to select the channel with largest channel coefficient even there is a (small) risk to pick an overloaded AP with load greater than β. Therefore, association protocol "max SNR" outperforms "min load" and performs extremely close to protocol "max throughput" which is optimal with respect to the average throughput. Even for small number of APs ($N = 4$), "max SNR" protocol doubles the throughput compared to the random association scheme. Gains with "min load" protocol also increase as a function of the number of APs with typical values above 30% for 4 APs for a large range of average load β. At high SNR, most of the channels have sufficient SNR in order to guarantee reliable transmission. Therefore, the throughput gains with "max SNR" are little compared to those achieved with "min load" protocol.

In order to validate the theoretical results of Section III, we evaluate the performance of the four protocols through Matlab-based Monte-Carlo simulations. For each run (over a total number of runs of 10000), we randomly uniformly associate βN users to N APs. The links between the newcoming terminal and the access points are modeled as flat fading channels with a fading coefficient $a_i[n]$. The coefficients $a_i[n]$ are redrawn each run from an i.i.d. complex Gaussian generator with zero mean and unit variance. The SNR for link $i,\ i = 1, \ldots, N$ can be expressed as $\text{SNR}_i = |a_i|^2/\sigma^2$ with σ^2 noise variance. The results are averaged over all runs. We assume that the terminal transmits with the optimal rate (4). For random association, we select in each run the first AP independently of its load or SNR. The "max SNR" protocol selects the AP with best SNR, i.e., the AP with largest value $|a_i|^2,\ i = 1, \ldots, N$. For low SNR, we consider $\sigma^2 = 1$ such that the average SNR over all links is equal to 0 decibels and for high SNR, we consider $\sigma^2 = 10^{-3}$ which gives an average SNR of 30 decibels. The "min load" protocol associates TS with AP with lowest load (if several APs have the same lowest load, the protocol picks one among those at random) and "max throughput" selects AP such that its link with TS maximizes (4). Surprisingly, simulation results match well with the theoretical results even for the case of small number of APs. This is important since it shows the benefit of our analysis for IEEE 802.11 systems for which only a few APs working on orthogonal channels are available (3 or 4 typically). The protocols "max SNR" and "max throughput" perform rather poorly in simulation compared to the theoretical ones in low SNR regime. It is due to the fact that the average low SNR considered in our simulation setup is equal to 0 decibels for which the approximation in (8) is not accurate. However, much lower transmission SNR $\ll 0$ decibel for which the approximation in (8) would be more accurate, is highly impractical and is not presented here. Equation (8) is still interesting since it provides an upper bound of the achievable gain for "max SNR" protocol in the low SNR regime.

B. IEEE 802.11 wireless LAN case study

All results presented above hold for Gaussian inputs. In this section, we propose to compare, by means of simulations, performance of all four protocols in a more realistic scenario. We consider an OFDM-based system similar to IEEE 802.11g. For all four protocols, all transmission parameters at the system level including the associated AP index are optimized by using genetic algorithm (GA). GA has been proven to be a very efficient method to solve non-linear problems [10]. The transmission parameters that we optimize in this paper are: the transmit power and bit load for each subcarrier, the minimum contention window size $CW_{\min}$, the maximum backoff stage m and the AP index to be associated. Performance results are compared in Fig. 3 against the results obtained with random association protocol. We assume that βN users are uniformly distributed among the APs with $\beta = 16$ and $N \in \{2, 4, 8\}$. It means that the current number of users among APs may significantly vary around average value $\beta = 16$. At low SNR (10 decibels), the best strategy consists in selecting the AP with the best transmission SNR. Although the potential high load of the AP may significantly reduce the throughput, selection based on the best SNR is still the optimal strategy which efficiently mitigates deep channel fading. As we have theoretically shown in Section III, the average throughput gain grows logarithmically with respect to the number of reachable APs at low SNR. However, the simulated throughput gains are smaller than the theoretically estimated gains. It is mainly due to the fact that in both random access and "max SNR" protocols, the minimum contention window size $CW_{\min}$ and the maximum backoff stage m are optimized (through GA). This optimization enhances the performance of both protocols equivalently and therefore reduces their overall performance gap. At high SNR, protocol "min load" outperforms the max SNR-based strategy. Indeed, almost all sender-AP links have large SNRs to guarantee reliable transmission with highest modulation order, 64 in our case. The throughput is then maximized for the AP with the smallest number of active users. At high SNR, GA-based "max throughput" protocol performs slightly worse than "min load" protocol. This behavior can be explained as follows. At high SNR (30 decibels), we use the same set of internal parameters for GA as in low the SNR regime in order to simplify the implementation. In that case, GA favors too much the transmission reliability at the expense of the throughput. Better performance would be attained if the internal parameters of GA are set with respect to the current transmission SNR. However, even if the "max throughput" protocol performs slightly worse than "min load" at high SNR, it outperforms the "max SNR" protocol. Therefore, our approach offers a good tradeoff between performance and implementation complexity.

V. Conclusion

In this paper, we have analyzed performance of three opportunistic frequency-agile multi-channel MAC protocols that associate a newcoming terminal station to one of the access points that are available within its transmission range. All three protocols achieve tremendous gain in an opportunistic and distributed manner for a limited amount of channel state information at the transmitter and that, even for a small number of access points. Based on our theoretical analysis, at high SNR and for 4 APs that are working on orthogonal channels, "min load" achieves at least 20% throughput gains whereas at low SNR the "max SNR" protocol doubles the throughput compared to the basic random association scheme. In our OFDM-based IEEE 802.11g simulation model, gains are 30% for "min load" at high SNR (30 decibels), 70% for "max SNR" at low SNR (10 decibels).

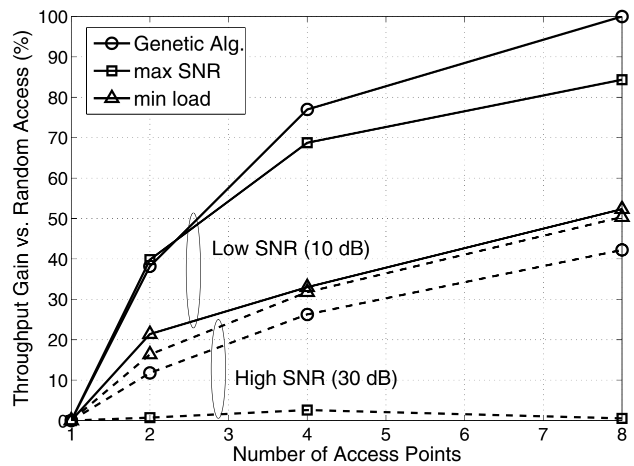

Fig. 3. Cross-Layer Optimization for Frequency-Agile Multi-Channel OFDM-based CSMA/CA system: Joint Power Optimization / Bit-loading Algorithm / Minimum Contention Window Size / Maximum Backoff Stage / Access Point Association. Target PER $= 10^{-2}$. Average System Load per Access Point $= 16$.

Acknowledgments

This work has been in part supported by DFG (Deutsche Forschungsgemeinschaft) and RWTH Aachen University through the UMIC excellence cluster funding.

References

[1] L. Zheng and D. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Transactions on Information Theory*, vol. 49, pp. 1073–1096, May 2003.

[2] D. Zheng and J. Zhang, "Protocol design and throughput analysis of frequency-agile multi-channel medium access control," *IEEE Transactions on Wireless Communications*, vol. 5, pp. 2887–2895, Oct. 2006.

[3] G. Bianchi, "IEEE 802.11-saturation throughput analysis," *IEEE Communications Letters*, vol. 2, pp. 318 – 320, Dec. 1998.

[4] S. Ci and H. Sharif, "Adaptive approaches to enhance throughput of IEEE 802.11 wireless LAN with bursty channel," in *Proceedings of the IEEE Conference on Local Computer Networks*, pp. 44–45, Nov. 2000.

[5] B.-S. Kim, Y. Fang, T. Wong, and Y. Kwon, "Throughput Enhancement Through Dynamic Fragmentation in Wireless LANs," *IEEE Transactions on Vehicular Technology*, vol. 54, pp. 1415–1425, July 2005.

[6] G. Bianchi, "Performance analysis of the IEEE 802.11 distributed coordination function," *IEEE Journal on Selected Areas in Communications*, vol. 18, pp. 535–547, March 2000.

[7] W. Lee, "Estimate of channel capacity in rayleigh fading environment," *IEEE Transactions on Vehicular Technology*, vol. 39, pp. 187–189, Aug. 1990.

[8] A. de Baynast, L. Wu, and P. Mähönen, "Throughput analysis for frequency-agile multi-channel medium access control," Tech. Rep. 2007-09_01, Department of Wireless Networks, RWTH Aachen University, Sep. 2007. Available: http://www.mobnets.rwth-aachen.de/private/ade/TR-rwth-aachen-mobnets-2007-09_01.pdf.

[9] P. Berenbrink, A. Czumaj, A. Steger, and B. Vöcking, "Balanced allocations: The heavily loaded case," *SIAM Journal of Computing*, vol. 35, pp. 1350–1385, June 2006.

[10] D. Whitley, "A Genetic Algorithm Tutorial," *Statistics and Computing*, vol. 4, pp. 65 – 85, 1994.